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Asymptotic Agreement Among Communicating Decisionmakers

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This paper studies asymptotic agreement among communicating decision-makers in terms of the evolution of a dynamical system defined on the lattice of information σ-algebras. This approach focuses on the concept of decisions based on common knowledge introduced earlier by Aumann, but it extends the investigation to general decision rules. We obtain conditions for asymptotic agreement in cases of direct, indirect, and random communications. We also present several examples to illustrate disagreement when the agreement conditions are not satisfied.

1. INTRODUCTION

Systems with decentralized information and limited communication are in general very difficult to analyze; their formulation so far has led to infinite dimensional, nonconvex, nonlinear optimization problems as evidenced by [1–8]. Even for the simplest problem with decentralized information and limited communication, it is very hard to compute the optimal performance [1] or even tight bounds on the performance [5, 6].

In this paper we analyze the asymptotic behavior of a simple system of distributed decisionmakers with limited communication. We do not attempt to prescribe optimal decision and communication strategies, but we instead analyze the consequences of given, plausible strategies. We believe that such analysis improves the understanding of what are good strategies and what tractable problem formulations may be.

The system considered in this paper consists of a finite number of decisionmakers who take measurements and have to decide among a finite or infinite number of hypotheses. Periodically, the decisionmakers can communicate their current decisions and then recompute their own decision. This process results in a sequence of decisions for each decisionmaker. We study several questions related to the evolution of these sequences of decisions: will an individual decisionmaker settle on a final decision; will all decisionmakers eventually agree on one hypothesis; if the decisionmakers eventually agree, on what will they agree?

The general problem of reaching a consensus of opinion among several decisionmakers has been studied by several statisticians and mathematicians (see [9] for background). In this paper we follow the recent work of [10]–[14] on the agreement problem, but we have tried to obtain a deeper understanding of the role of information in the evolution of decisions. In order to do this we have described the decisionmakers' information in terms of a dynamical system evolving in the lattice of information σ-algebras. Abstract lattice dynamical systems have been studied and applied to other information processing problems in [15]. In our formulation the state of the dynamical system is an n-tuple of σ-algebras describing each decisionmaker's current information. Note that Witsenhausen introduced the idea of using a σ-algebra as the state of a stochastic system in [16, 17]. Of crucial importance in the formulation proposed here is the idea of "common information" or "common knowledge" which determines the outcome of the agreement in our problem. This notion was defined in [10, 18, 19], and our work takes a similar approach in this respect.

With this approach we have been able to characterize general decision rules for which asymptotic agreement among communicating decisionmakers is possible. Thus, our results contain those of [10–14] as special cases.

The paper is organized as follows. In Section 2 we formulate the agreement problem precisely in terms of a lattice dynamical system defined on the lattice of information σ-algebras. We consider the simple case in which each decisionmaker communicates directly with each other and in which the number of different measurements is finite. We relate our results to the earlier work of [10, 11] and the notion of "common knowledge" defined in [10]. In Section 3 we generalize the problem of Section 2 to allow an infinite number of different measurements. Under certain conditions there is eventual agreement, and we relate these conditions to the results of [12, 13]. We also present several examples to show what can happen when these conditions are not satisfied. In Section 4 we generalize the problem formulation of Sections 2 and 3 to permit indirect communication. That is, there is not necessarily a communication link between every pair of decisionmakers. We compare these results to [12, 13]. Section 5 formulates random communication (delays and distortion) in the lattice σ-algebra framework. In this framework we can obtain results similar to [12, 13]. Section 6 concludes our report.

2. AGREEMENT AMONG DECISIONMAKERS WITH FINITE MEASUREMENTS AND DIRECT COMMUNICATION

2.1 Introduction

In this section we study the problem of agreement among decisionmakers with finite measurements and direct communications. We determine the conditions under which the decisionmakers eventually agree after a finite number of communications. The main conceptual conditions are that: (1) each decisionmaker uses the same rule (with the same prior information) for generating his decision, and (2) the resulting decision conveys enough information to reconstruct the decision (using the given rule). These conditions are formulated precisely in subsections 2.2 and 2.3. As an example, we will discuss in subsection 2.4 the algorithm corresponding to the maximum a posteriori (MAP) decision rule. In subsection 2.5 we interpret condition 2 in terms of the notion of "common knowledge" introduced in [10].

2.2 Lattice dynamical system formulation

We assume a stochastic decisionmaking model throughout this paper. The underlying probability space is denoted $(\Omega, \mathscr{F}, P)$ where Ω is the sample space, $\mathscr{F}$ is the σ-algebra of events, and P is the probability measure. The hypothesis is represented by a random function $x:\Omega \rightarrow X$ taking values in a finite set X. Decisionmaker k takes measurements in a set Y_k. The measurements are represented by random functions $y_k:\Omega\to Y_k$. The main technical assumption in this section is that each Y_k is a finite set. We study what happens with infinite Y_k in Section 3. If f is a measurable mapping of Ω, $\mathscr{F}$ into some measure space, let $\sigma(f)$ denote the σ-algebra in $\mathscr{F}$ generated by f. In the case of $f=y_k$, let us denote $\sigma(y_k)$ by $\mathscr{Y}_k$. For our purposes, only the σ-algebras $\mathscr{Y}_k$ are relevant, and we need not consider y_k, Y_k. Note that since Y_k is finite, so is $\mathscr{Y}_k$.

In this paper we represent information by σ-algebras contained in $\mathscr{F}$. A *decision rule* is a function d which maps σ-algebras $\mathscr{G}$ to *decision functions* $d(\mathscr{G}):\Omega\to X$. *We assume that $d(\mathscr{G})$ is $\mathscr{G}$-measurable.* Note that this means $\sigma(d(\mathscr{G}))\subset\mathscr{G}$. Let $\mathscr{F}_k(t)$ denote the information σ-algebra of decisionmaker k just after the t-th $(t\geqq 0)$ communication. The decision made on the basis of this information is the value of the function $d(\mathscr{F}_k(t))$, and this decision is communicated to the other agents. The σ-algebras evolve dynamically as follows

$$\mathscr{F}_k(t+1)=\mathscr{F}_k(t)\vee\bigvee_{j\neq k}\sigma(d(\mathscr{F}_k(t))) \tag{2.1}$$

with the initial condition

$$\mathscr{F}_k(0)=\mathscr{Y}_k. \tag{2.2}$$

Note that $\vee$ is the *join* operation on σ-algebras: $\mathscr{G}_1\vee\mathscr{G}_2$ is the smallest σ-algebra containing $\mathscr{G}_1$ and $\mathscr{G}_2$.

Knowing the mapping d and the dynamic equations (Eqs. 2.1 and 2.2) tells us everything we need to know about the evolution of information and decisions in this problem. We may view Eqs. 2.1 and 2.2 as a dynamical system defined on the lattice of σ-algebras contained in $\mathscr{F}$. The lattice operations are $\vee$ (as defined above) and $\cap$ (set intersection). The maximum σ-algebra is $\mathscr{F}$ and the minimum one is the trivial σ-algebra $\mathscr{F}_0:=\{\varnothing,\Omega\}$. Equation 2.1 generates increasing sequences $\{\mathscr{F}_k(t)\}_{t\geqq 0}$ of σ-algebras. We are interested in the evolution of $\mathscr{F}_k(t)$ and the corresponding decision functions $d(\mathscr{F}_k(t))$ as $t\to\infty$.

2.3 Condition for agreement

The conceptual condition 2 stated in subsection 2.1 is made precise in the following proposition.

PROPOSITION 2.1 *Suppose that d has the property that whenever $\sigma(d(\mathscr{G}_1)) \subset \mathscr{G}_2 \subset \mathscr{G}_1$ then $d(\mathscr{G}_1) = d(\mathscr{G}_2)$. If $\{\mathscr{F}_k(t)\}_{t \geqq 0}$ satisfy Eqs. (2.1) and (2.2), then there are σ-algebras $\mathscr{F}_k \subset \bigvee_j \mathscr{Y}_j$ and an integer $T \geqq 0$ such that*

$$\mathscr{F}_k(t) = \mathscr{F}_k \tag{2.3}$$

for $t \geqq T$, and

$$d(\mathscr{F}_k) = d(\mathscr{F}_j) = d\left(\bigcap_i \mathscr{F}_i\right) \tag{2.4}$$

for all j, k.

Proof By the finiteness assumption we know that $\bigvee_j \mathscr{Y}_j$ is finite. For each $t \geqq 0$,

$$\mathscr{Y}_k \subset \mathscr{F}_k(t) \subset \mathscr{F}_k(t+1) \subset \bigvee_j \mathscr{Y}_j.$$

It follows that eventually $\mathscr{F}_k(t) = \mathscr{F}_k$ for all $t \geqq T$ for some $T \geqq 0$. Equation 2.1 for $t \geqq T$ says that

$$\mathscr{F}_k = \mathscr{F}_k \vee \bigvee_{j \neq k} \sigma(d(\mathscr{F}_j))$$

or equivalently,

$$\sigma(d(\mathscr{F}_j)) \subset \mathscr{F}_k$$

for all j, k. Thus, we have

$$\sigma(d(\mathscr{F}_j)) \subset \bigcap_i \mathscr{F}_i \subset \mathscr{F}_j$$

and therefore (by the assumed property of d)

$$d(\mathscr{F}_j) = d\left(\bigcap_i \mathscr{F}_i\right)$$

for each j. This completes the proof.

2.4 EXAMPLE

To illustrate this approach let us consider the decision algorithm d defined by the MAP decision rule. That is, if $\mathscr{G}$ is a σ-algebra, then let $d(\mathscr{G})$ be defined as

$$d(\mathscr{G}) = \arg\max \{P(x=\xi|\mathscr{G})|\xi \in X\}. \tag{2.5}$$

The arg max function is defined precisely as follows. Let $\{\lambda_\xi|\xi \in X\}$ be a collection of real numbers; and let $>$ be a total order relation on X. Then

$$u = \arg\max \{\lambda_\xi|\xi \in X\}$$

if and only if

$$\lambda_u \geqq \lambda_\xi$$

for all $\xi \in X$, and if $\lambda_u = \lambda_\xi$ then

$$u > \xi.$$

Note that the arg max depends on $>$ to break ties consistently in case $\{\lambda_\xi : \xi \in X\}$ does not have a unique maximum. Now we show that d defined in Eq. 2.5 satisfies the property described in Proposition 2.1.

PROPOSITION 2.2 *Suppose d is defined by Eq. (2.5) and G_1, $\mathscr{G}_2$ are σ-algebras such that $\sigma(d(\mathscr{G}_1)) \subset \mathscr{G}_2 \subset \mathscr{G}_1$. Then $d(\mathscr{G}_1) = d(\mathscr{G}_2)$.*

Proof Consider the event $E_u = \{u = d(\mathscr{G}_1\}$. Note that $E_u \in \sigma(d(\mathscr{G}_1))$ and $E_u \in \mathscr{G}_2$ by assumption. By definition Eq. (2.5) of d,

$$P(x=u|\mathscr{G}_1) \geqq P(x=\xi|\mathscr{G}_1) \quad \text{a.s.} \quad E_u\dagger$$

for all $\xi \in X$. Conditioning this inequality with respect to $\mathscr{G}_2$ (noting

†"a.s. E_u" = "for almost all samples ω in E_u." Note that ω is the unexpressed argument of all conditional probabilities and expectations.

$E_u \in \mathscr{G}_2$) gives

$$P(x=u|\mathscr{G}_2) \geqq P(x=\xi|\mathscr{G}_2) \quad \text{a.s.} \quad E_u.$$

Suppose equality holds. Then we have

$$E(1_{E_u} \cdot (P(x=u|\mathscr{G}_1) - P(x=\xi|\mathscr{G}_1))|\mathscr{G}_2) = 0 \tag{2.6}$$

where 1_{E_u} denotes the indicator function, $1_{E_u}(\omega)=0$ if $\omega \neq E_u$ and $1_{E_u}(\omega)=1$ if $\omega \in E_u$. Since

$$1_{E_u} \cdot (P(x=n|\mathscr{G}_1) - P(x=\xi|\mathscr{G}_1)) \geqq 0 \quad \text{a.s.} \quad ,$$

it follows from Eq. (2.6) that in fact

$$1_{E_u} \cdot (P(x=u|\mathscr{G}_1) - P(x=\xi|\mathscr{G}_1)) = 0 \quad \text{a.s.} \quad .$$

Since u is the arg max $\{P(x=\xi|\mathscr{G}_1\}$ on E_u, we see that $u > \xi$. Thus, we see that

$$u = \arg\max \{P(x=\xi|\mathscr{G}_2)\} = :d(\mathscr{G}_2)$$

on E_u. It follows that $d(\mathscr{G}_1) = d(\mathscr{G}_2)$.

Note that the proof of Proposition 2.2 is true even if the σ-algebras $\mathscr{G}_1$, $\mathscr{G}_2$ are infinite.

Note that MAP is not the only decision rule that satisfies the property in Proposition 2.1. Suppose we allow the space of decisions to be infinite. The property of Proposition 2.1 will make sense and Proposition 2.1 will still be true as long as the measurements $\mathscr{Y}_k$ are finite. For example, $d(\mathscr{G})$ may be $E(x|\mathscr{G})$, the conditional expectation of some random variable, or $d(\mathscr{G})$ may be a vector with components $P(x=\xi|\mathscr{G})$ where $\xi \in X$.

2.5 Common knowledge

In [10] information is represented by a partition of the sample space Ω. There is an obvious correspondence between partitions and σ-algebras. Note that the *meet* $P_1 \wedge P_2$ of two partitions corresponds to the intersection $\mathscr{F}_1 \cap \mathscr{F}_2$ of the corresponding σ-algebras. Aumann [10] defines an event E to be *common knowledge to* *decisionmakers 1 and 2 (with information $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively)* at ω if there is an atom $F \in \mathscr{P}_1 \wedge \mathscr{P}_2$ such that $\omega \in F \subset E$. An equivalent definition in terms of the σ-algebras $\mathscr{F}_1$ and $\mathscr{F}_2$ corresponding to $\mathscr{P}_1$ and $\mathscr{P}_2$ is that there is an event $F \in \mathscr{F}_1 \cap \mathscr{F}_2$ such that $\omega \in F \subset E$. Let us say that the event is common knowledge to decisionmakers 1 and 2 if it is common knowledge at each $\omega \in W$. Then E is common knowledge to 1 and 2 if and only if it belongs to the σ-algebra generated by $\mathscr{P}_1 \wedge \mathscr{P}_2$, namely $\mathscr{F}_1 \cap \mathscr{F}_2$.

In [10] Aumann noted the simple fact that if the conditional probability functions $P(E|\mathscr{F}_1)$ and $P(E|\mathscr{F}_2)$ are common knowledge to 1 and 2 (i.e., both are $\mathscr{F}_1 \cap \mathscr{F}_2$ measurable), then $P(E|\mathscr{F}_1) = P(E|\mathscr{F}_2)$. Let us compare this to the condition for agreement in subsection 2.3. We have the following proposition whose proof follows directly from the definitions.

PROPOSITION 2.3 *The conditions*

$$\sigma d(\mathscr{F}_1) \subset \mathscr{F}_2 \subset \mathscr{F}_1 \Rightarrow d(\mathscr{F}_1) = d(\mathscr{F}_2) \tag{2.7}$$

and

$$\sigma d(\mathscr{F}_1) \vee \sigma d(\mathscr{F}_2) \subset \mathscr{F}_1 \cap \mathscr{F}_2 \Rightarrow d(\mathscr{F}_1) = d(\mathscr{F}_2) \tag{2.8}$$

are equivalent.

In Aumann's language [10] Eq. (2.8) asserts that if decisions $d(\mathscr{F}_1)$ and $d(\mathscr{F}_2)$ are common knowledge to 1 and 2, then they are equal. Aumann showed this condition is satisfied by the decision rule

$$d(\mathscr{F}) = P(E|\mathscr{F}).$$

We saw that this condition is satisfied by the MAP decision rule also. In Section 4 we will characterize the class of decision rules d which satisfy a stricter version of this condition.

Expressed in terms of common knowledge, Proposition 2.1 states that if all decisionmakers use the same decision rule, if the decisionmakers communicate simultaneously, and if common knowledge decisions agree, then all decisionmakers agree on the same decision after a finite number of communications. The common decision is the decision based on the ultimate common knowledge of the decisionmakers. With this interpretation, Proposition 2.1 generalizes the result of [11] which considered the case in which $d(\mathscr{F})$ was the posterior probability $P(E|\mathscr{F})$ given $\mathscr{F}$.

3. AGREEMENT AMONG DECISIONMAKERS WITH INFINITE MEASUREMENTS AND DIRECT COMMUNICATION

3.1 Introduction

The problem formulation of the previous section does not require that the information σ-algebras be finite. One can generalize the formulation of the previous section to allow infinite measurement σ-algebras and an infinite decision space. To do this one needs to interpret agreement in terms of the convergence of a sequence of decision functions. In this paper we will assume that the decision space is a metric space, and convergence will always mean almost sure convergence with respect to that metric. We will also assume that all σ-algebras are complete with respect to the underlying probability measure. This ensures that limits of decision functions are measurable. The difficulty with infinite measurements is that the σ-algebras $\mathscr{F}_k(t)$ do not necessarily converge to the limit $\mathscr{F}_k = \bigvee_{t \geq 0} \mathscr{F}_k(t)$ after a finite number of communications. As we will see in the examples below, it is possible that d satisfies the conditions of Proposition 2.1 and $d(\mathscr{F}_k(t))$ does not converge or does not converge to $d(\mathscr{F}_k)$. In the next subsection we show that the agreement results of Section 2 are true provided that d satisfies a continuity condition. Not all interesting decision rules d satisfy this continuity condition. Decision rules which give discrete-valued decision functions (e.g., the MAP rule of Section 2) are examples of rules not satisfying the continuity condition. We give examples and discuss this problem later in the section.

3.2 Continuity condition for agreement

The following proposition defines continuity for decision rules d. Together with the earlier condition in Proposition 2.1, this condition allows us to prove asymptotic agreement among decisionmakers who take infinitely many measurements and communicate directly with each other.

PROPOSITION 3.1 *Suppose that a decision rule d satisfies the following two conditions:* (1) *if* $\sigma d(\mathscr{G}_1) \subset \mathscr{G}_2 \subset \mathscr{G}_1$, *then* $d(\mathscr{G}_1) = d(\mathscr{G}_2)$; (2) *if* $\mathscr{G}_n \subset \mathscr{G}_{n+1}$ *and* $\mathscr{G} = \bigvee_{n \geqq 1} \mathscr{G}_n$, *then* $\lim_{n \to \infty} d(\mathscr{G}_n) = d(\mathscr{G})$. *If* $\{\mathscr{F}_k(t)\}_{t \geqq 0}$ *satisfies Eqs.* (2.1) *and* (2.2) *and if* $\mathscr{F}_k = \bigvee_{t \geqq 0} \mathscr{F}_k(t)$, *then*

$$\lim_{t \to \infty} d(\mathscr{F}_k(t)) = d(\mathscr{F}_k) \tag{3.1}$$

$$d(\mathscr{F}_k) = d\left(\bigcap_j \mathscr{F}_j\right) \tag{3.2}$$

for each k.

Proof Since $\mathscr{F}_k(t) \uparrow \mathscr{F}_k$, assumption 2 implies that Eq. (3.1) is true. Thus, from Eq. (2.1), for each k, j we have

$$\sigma d(\mathscr{F}_k(t)) \subset \mathscr{F}_j.$$

By the assumption concerning convergence in the preceding paragraph, we see that

$$\sigma d(\mathscr{F}_k) \subset \mathscr{F}_j$$

for each j, k. The result (Eq. 3.2) follows from condition 1.

Note that the same proof and result are true for the case in which the decisionmakers can take measurements at each time.

If the decision space and the measurement σ-algebras are finite (as was assumed in Section 2), then every decision function automatically satisfies the continuity condition defined with respect to the discrete metric (i.e., the metric that assigns distance 0 between point and itself, and distance 1 between a point and any other point in the metric space). If the decision space is finite but the measurement σ-algebras are not, then the continuity condition may not be true as we show in the examples below.

The continuity condition does hold in many cases in which the decision space is infinite. For example, the continuity condition holds in the case in which d is defined as a conditional expectation

$$d(\mathscr{F}) = E(x \mid \mathscr{F})$$

for some random variable (or finite dimensional vector) x. Condition 2 in this case is just the martingale convergence property [20]. Proposition 3.1 for this specific decision rule is included in the more general results of [12]. In [13] more general decision rules satisfying 2 are defined in the following way. Suppose $J:Dx\Omega \to R$ is a cost function and $d(\mathscr{F})$ minimizes $J(f(\cdot), \cdot)$ for all $\mathscr{F}$ measurable decision functions $f:\Omega \to D$. If $\mathscr{G}_n \uparrow \mathscr{G}$, then under fairly weak conditions $E(J(d(\mathscr{G}_n), \cdot)|\mathscr{G}_n)$ is a convergent supermartingale [20] and it converges to $E(J(d(\mathscr{G}), \cdot)|\mathscr{G})$. Under some conditions (e.g., uniform strict convexity of J, where decisions take values in a finite dimensional compact, convex set [13]), the convergence of the optimal costs implies the convergence of the arguments $d(\mathscr{G}_n) \to d(\mathscr{G})$. Note that these decision rules also satisfy condition 1. We will examine this further in Section 4.

3.3 Examples

It is interesting to note that some decision rules (the MAP rule, for example) do not satisfy 2 but nevertheless satisfy the conclusion of Proposition 2.1 in some cases. In other cases asymptotic agreement is not achieved, and it is an open problem to determine useful conditions for agreement in such cases. The following examples illustrate possible behavior of the MAP decision rule.

Example 3.1 The first example shows that the MAP rule defined in subsection 2.4 need not satisfy the continuity condition 2 of Proposition 3.1. Let the sample space by the unit square $[0,1]^2$, let the events be Borel sets, and let the probability P be uniformly distributed over the unit square. For any real number r_1 and r_2, let $E(r_1, r_2)$ denote the subset of $[0,1]^2$ defined by

$$E(r_1, r_2) = \{(\omega_1, \omega_2) \in [0,1]^2 = r_2 \geqq \omega_2 - \omega_1 > r_1\}$$

Let $E_0 = E(1/\sqrt{2}, 1)$ and $E_1 = E(-1, -1/\sqrt{2})$.

Note that $P(E_0) = P(E_1) = 1/4$. Given positive integers n, m, one can choose $n+m+1$ real numbers r_k such that $r_1 > r_2 > \ldots > r_{n+m+1}$, $r_1 = 1/\sqrt{2}$, $r_{n+1} = 0$, $r_{n+m+1} = -1/\sqrt{2}$, and such that $P(E(r_{k+1}, r_k))$ is $1/4n$ for $1 \leqq k \leqq n$ and is $1/4m$ for $n+1 \leqq k \leqq n+m$.

Define $\mathscr{G}_{n,m}$ to be the σ-algebra generated by the events

$E(r_{k+1}, r_k)$ for $2 \leqq k \leqq m+n-1$, and the event $E_0 \cup E_1 \cup E(r_2, r_1) \cup E(r_{m+n+1}, r_{m+n})$. Define the random function $x:[0,1]^2 \to \{0,1\}$ as $x(\omega)=0$ for $\omega \in E(0,1)$ and $x(\omega)=1$ for $\omega \in E(-1,0)$. Let d be the MAP decision rule with respect to the hypotheses $x=0$ and $x=1$. Assume that d prefers 0 in case of ties. It is easy to compute that if $\omega \in E_0 \cup E_1$, then

$$d(\mathscr{G}_{n,m}) = 0 \quad \text{if} \quad n \leqq m$$

$$= 1 \quad \text{if} \quad n > m.$$

Furthermore, if $\mathscr{G} = \bigvee_{n,m} \mathscr{G}_{n,m}$, then if $\omega \in E_0 \cup E_1$,

$$D(\mathscr{G}) = 0.$$

Thus, if $\mathscr{G}_{2k} = \mathscr{G}_{k,k}$, $\mathscr{G}_{2k+1} = \mathscr{G}_{k+1,k}$, we see that for $\omega \in E_0 \cup E_1$

$$\lim_{k \to \infty} d(\mathscr{G}_{2k}) = 0 = d(\mathscr{G})$$

$$\lim_{k \to \infty} d(\mathscr{G}_{2k+1}) = 1 \neq 0 = d(\mathscr{G})$$

and $\lim_{k \to \infty} d(\mathscr{G}_k)$ does not exist.

Example 3.2 It is possible that asymptotic agreement occurs even though the decision rule does not satisfy the continuity condition. This is the case for the previous example when there are two decisionmakers with respective measurements $y_1(\omega_1, \omega_2) = \omega_1$ and $y_2(\omega_1, \omega_2) = \omega_2$. In this case decisionmaker 1 communicates the binary expansion of ω_1 bit-by-bit up to the time of agreement. Similarly, decisionmaker 2 communicates the binary expansion of $1-\omega_2$ up to the time of agreement. The proof of this is straightforward but lengthy, so we omit it and refer the reader to [22]. The hypothesis $x=0$ is the event that $\omega_2 > \omega_1$ and $x=1$ is the event that $\omega_2 \leqq \omega_1$. Thus, if $\omega_1 \neq \omega_2$, these communications enable each decisionmaker to determine x with certainty after a finite number of communications. If $\omega_1 = \omega_2$ and ω_1 is a binary rational number, then the two decisionmakers also agree after a finite number of communications. However, if $\omega_1 = \omega_2$ and ω_1 is an irrational number, then the two decisionmakers never agree, nor do their decisions ever converge. However, the probability that $\omega_1 = \omega_2$ is 0, and thus, the two decisionmakers agree almost surely after a finite number of communications.

Example 3.3 A simple variation of the previous two examples shows that asymptotic agreement does not occur always. Define the sample space Ω to be the union

$$\Omega = \{(\omega_1, \omega_2, 1){:}0 \leqq \omega_1 \leqq \omega_2 \leqq 1\} \cup \{(\omega_1, \omega_2, 0){:}0 \leqq \omega_2 \leqq \omega_1 \leqq 1\}.$$

This sample space is the same as that of Example 3.2 except that we have added a copy of the diagonal $\omega_1 = \omega_2$. Let P_0 be the probability distributed uniformly on the diagonal $\{(\omega, \omega, 0){:}0 \leqq \omega \leqq 1\}$ and let P_1 be the probability distributed uniformly on the other diagonal $\{(\omega, \omega, 1){:}0 \leqq \omega \leqq 1\}$. Let P_2 be the probability distributed uniformly on the rest of Ω. Define the probability P on Ω as

$$P = \alpha_0 \cdot P_0 + \alpha_1 \cdot P_1 + \alpha_2 \cdot P_2$$

where $\alpha_0 + \alpha_1 + \alpha_2 = 1$ and $0 \leqq \alpha_0, \alpha_1, \alpha_2$. Define $y_k(\omega_1, \omega_2, \omega_3) = \omega_k$, $k = 1$ or 2, and let H_j be the hypothesis $\omega_3 = j$. If $\alpha_0 + \alpha_1 = 0$, then the decision functions depend only on ω_1 and ω_2 and are the same as for Example 3.2. However, if $1/2 > \alpha_0 = \alpha_1 > 0$, then the diagonals have positive probability $\alpha_0 + \alpha_1 > 0$. Indeed, the decision-makers will never agree on the points $(\omega, \omega, \omega_3)$ if ω is not a binary rational number. Such sample points have a total probability equal to the probability of the diagonal, namely $\alpha_0 + \alpha_1 > 0$. Thus, the decisionmakers never agree with positive $(\alpha_0 + \alpha_1)$ probability.

4. AGREEMENT AMONG DECISIONMAKERS WITH INDIRECT COMMUNICATIONS

4.1 Introduction

In the previous two sections we assumed that all decisionmakers communicated directly with each other. In this section we examine the agreement problem in cases where some pairs of decisionmakers do not communicate directly, but communicate indirectly through other decisionmakers. In such cases of indirect communication the agreement condition of Section 2 is not sufficient. In this section we show that a stronger agreement condition is necessary and sufficient to guarantee asymptotic agreement if only indirect communication is assumed. We relate this condition to the results of [12] and [13], and we show that the new agreement condition is equivalent to requiring that the decision rule is optimal with respect to some preference relation among decisions.

4.2 Agreement condition for rings of decisionmakers

We say that a decision rule d satisfies the *agreement condition for rings of decisionmakers* if $\sigma d(\mathscr{F}_k) \subset \mathscr{F}_{k+1}$ for $k=1,2,\ldots,n$ and $\sigma d(\mathscr{F}_{n+1}) \subset \mathscr{F}_1$, then $d(\mathscr{F}_k) = d(\mathscr{F}_1)$ for all k. Note that the original agreement condition in sub-section 2.3 is a condition for *pairs of decisionmakers.* We now show that the agreement condition for rings is both necessary and sufficient for asymptotic agreement among decisionmakers who communicate indirectly.

Consider the lattice dynamical system

$$\mathscr{F}_k(t+1) = \mathscr{F}_k(t) \vee \bigvee_{j \in A_k} \sigma d(\mathscr{F}_j(t)) \tag{4.1}$$

where $k=1,2,\ldots,n$ and $A_k \subset \{1,2,\ldots,n\}$. The sets $\{A_k\}$ determine a *fixed communication pattern* between the decisionmakers $\{k\}$. In Section 5 we will see how to treat time-varying and random communication patterns in the framework of lattice dynamical systems. Let us say here that a decisionmaker k_1 *communicates to* k_2 ($k_1 \to k_2$) if there is a sequence $k(j)$, $1 \leqq j \leqq N$, of decisionmakers such that $k_1 = k(1)$, $k_2 = k(N)$, and $k(j) \in A_k(j+1)$ for each $j=1,2,\ldots,N-1$. We say that k_1 *and* k_2 *communicate indirectly* ($k_1 \leftrightarrow k_2$) if either $k_1 = k_2$ or if both $k_1 \to k_2$ and $k_2 \to k_1$. Note that $\leftrightarrow$ is an equivalence relation between decisionmakers. This equivalence relation partitions the set of decisionmakers into equivalence classes, denoted $[k]$, of decisionmakers j such that $j \leftrightarrow k$. Equivalently, $j \leftrightarrow k$ if and only if j and k belong to a "communicating ring" [12]. With this notation we can prove the following results.

PROPOSITION 4.1 *Suppose that d satisfies the agreement condition for rings of decisionmakers. If $\{\mathscr{F}_k(t)\}_{t \geqq 0}$, $1 \leqq k \leqq n$, satisfy Eq. (4.1) with the initial conditions $\mathscr{F}_k(0) = \mathscr{Y}_k$, and if $\bigvee_k \mathscr{Y}_k$ is a finite σ-algebra, then there are σ-algebras $\mathscr{F}_k \subset \bigvee_j Y_j$ and an integer $T \geqq 0$ such that*

$$\mathscr{F}_k(t) = \mathscr{F}_k \tag{4.2}$$

for $t \geqq T$, and

$$d(\mathscr{F}_k) = d(\mathscr{F}_j) = d\left(\bigcap_{l \in [k]} \mathscr{F}_d\right) \tag{4.3}$$

for all k, j such that $k \leftrightarrow j$ or equivalently, $j \in [k]$ (or $k \in [j]$).

Proof Arguing as in the proof of Proposition 2.1, we easily see that there exist $\mathscr{F}_k$ satisfying Eq. 4.2 and such that

$$\sigma d(\mathscr{F}_k) \subset \mathscr{F}_j \tag{4.4}$$

if $j \in A_k$. If $k_1 \leftrightarrow k_2$, then there are $m(i)$ such that $k_1 = m(i_1)$ for some i_1, $k_2 = m(i_2)$ for some i_2, $m(N+1) = m(1)$, and

$$m(j) \in A_m(j+1)$$

for each j. Hence, we have also

$$\sigma d(\mathscr{F}_{m(j+1)}) \subset \mathscr{F}_{m(j)} \tag{4.5}$$

for each j. The relationship (Eq. 4.5) together with the agreement condition for rings imply that $d(\mathscr{F}_{m(j)}) = d(\mathscr{F}_{m(1)})$. Thus, $k_1 \leftrightarrow k_2$ implies that $d(\mathscr{F}_{k_1}) = d(\mathscr{F}_{k_2})$. It follows that Eq. 4.3 is also true.

PROPOSITION 4.2 *Suppose that d satisfies the agreement condition for rings and the continuity condition of Section 3. If $\{\mathscr{F}_k(t)\}_{t \geqq 0}$, $1 \leqq k \leqq n$, satisfy Eq. (4.1) with the initial conditions $\mathscr{F}_k(0) = \mathscr{Y}_k$, and if $\mathscr{F}_k = \bigvee_{t \geqq 0} \mathscr{F}_k(t)$, then*

$$\lim_{t \to \infty} d(\mathscr{F}_k(t)) = d(\mathscr{F}_k) \tag{4.6}$$

$$d(\mathscr{F}_k) = d\left(\bigcap_{j \in [k]} \mathscr{F}_j\right) \tag{4.7}$$

for each k.

Proof The proof of Eq. (4.6) is the same as in Proposition 3.1. Likewise, the argument of Proposition 3.1 proves Eq. 4.4 in this case also. The result Eq. 4.7 then follows exactly as in the previous proposition.

It should be clear that the agreement condition for rings is necessary as well as sufficient for Proposition 4.1 to be true. The following example shows that the agreement condition for pairs of decisionmakers is generally weaker than the agreement condition for rings.

Example 4.1 The sample space for this example is $\{2,3,5,7\}$. Let $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$ be σ-algebras generated by the partitions $\{\{2\}, \{3\}, \{5,7\}\}$, $\{\{2\}, \{5\}, \{3,7\}\}$, and $\{\{3\}, \{5\}, \{2,7\}\}$ respectively. For any σ-algebra $\mathscr{G}$ of this sample space, there is a partition $\{E_1, E_2, \ldots, E_n\}$ which generates $\mathscr{G}$. If $\mathscr{G} \neq \mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3$, then define $d(\mathscr{G})$ as the function which takes as its value on E_k the product of the integers in E_k. Each product depends uniquely on E_k, and therefore $\sigma d(\mathscr{G}) = \mathscr{G}$ for $\mathscr{G} \neq \mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3$. Define $d(\mathscr{F}_1) = d(\mathscr{F}_1 \cap \mathscr{F}_3)$, $d(\mathscr{F}_2) = d(\mathscr{F}_1 \cap \mathscr{F}_2)$, and $d(\mathscr{F}_3) = d(\mathscr{F}_2 \cap \mathscr{F}_3)$. Thus, $\sigma d(\mathscr{F}_1) = \mathscr{F}_1 \cap \mathscr{F}_3$, $\sigma d(\mathscr{F}_2) = \mathscr{F}_1 \cap \mathscr{F}_2$, and $\sigma d(\mathscr{F}_3) = \mathscr{F}_2 \cap \mathscr{F}_3$. It is easy to check that d satisfies the agreement condition for direct communication as defined in Section 2. It does not, however, satisfy the agreement condition for rings. Although $\sigma d(\mathscr{F}_2) \subset \mathscr{F}_1$, $\sigma d(\mathscr{F}_3) \subset \mathscr{F}_2$, and $\sigma d(\mathscr{F}_1) \subset \mathscr{F}_3$, the decision functions $d(\mathscr{F}_1)$, $d(\mathscr{F}_2)$, and $d(\mathscr{F}_3)$ are all different.

Example 4.2 The example above shows that the agreement condition for pairs of decisionmakers is weaker than the condition for rings of decisionmakers. Nevertheless, most decision rules of interest satisfy the ring condition. We will examine such decision rules further in the next subsection. Here we note that the conditional expectation decision rule d, defined by $d(\mathscr{F}) = E(x|\mathscr{F})$ for an integrable random variable x, satisfies the agreement condition for rings. This is proved in [12].

4.3 Agreement condition for rings and optimal decision rules

The paper [13] extended the results of [12] by showing that decision rules which optimize a scalar cost function give asymptotic agreement. Note that the conditional expectation decision rule of [12] is a special case of this since the conditional expectation is optimal for quadratic cost functions. In this subsection we show that the agreement condition for rings is satisfied only by decision rules which are optimal in a sense we will define precisely below.

PROPOSITION 4.3 *Suppose that* d *is a decision rule such that* $\sigma d(\mathscr{F}) \subset \mathscr{F}$ *for all* σ*-algebras* $\mathscr{F}$ *which are sub-algebras of the* σ*-algebra* $\mathscr{F}'$ *of the sample space* Ω*. Then* d *satisfies the agreement condition for rings if and only if there is a partial ordering* $\leqq$ *of the set of functions* $\{d(\mathscr{F}):\mathscr{F} \subset \mathscr{F}'\}$ *such that* $d(\mathscr{F})$ *is the maximum element of* $\{d(\mathscr{G}):\mathscr{G} \subset \mathscr{F}', \sigma d(\mathscr{G}) \subset \mathscr{F}\}$ *with respect to* $\leqq$.

Proof We first show that if

$$d(\mathscr{F}) = \max \{d(\mathscr{G}):\mathscr{G} \subset \mathscr{F}', \sigma d(\mathscr{G}) \subset \mathscr{F}\} \tag{4.8}$$

then d satisfies the agreement condition for rings. Suppose that $\sigma d(\mathscr{F}) \subset \mathscr{G} \subset \mathscr{F}$. Because $\mathscr{G} \subset \mathscr{F}$, it is clear that $d(\mathscr{G}) \leqq d(\mathscr{F})$. Since $\sigma d(\mathscr{F}) \subset \mathscr{G}$, it is clear that $d(\mathscr{F}) \in \{d(\mathscr{H}):\mathscr{H} \subset \mathscr{F}', \sigma d(\mathscr{H}) \subset \mathscr{G}\}$. Thus, $d(\mathscr{F}) \leqq d(\mathscr{G})$. The relation $\leqq$ is a partial order, and therefore, $d(\mathscr{G}) \leqq d(\mathscr{F})$ and $d(\mathscr{F}) \leqq d(\mathscr{G})$ imply $d(\mathscr{F}) = d(\mathscr{G})$. Thus d satisfies the agreement condition for pairs and in particular, $d(\sigma d(\mathscr{F})) = d(\mathscr{F})$.

Suppose that $\mathscr{F}_1 = \mathscr{F}_{n+1}$ and $\sigma d(\mathscr{F}_{k+1}) \subset \mathscr{F}_{k+1}$ for $1 \leqq k \leqq n$. Then

$$d(\mathscr{F}_k) = d(\sigma d(\mathscr{F}_k)) \leqq d(\mathscr{F}_{k+1})$$

for each k, and hence $d(\mathscr{F}_1) \leqq d(\mathscr{F}_k) \leqq d(\mathscr{F}_1)$ and therefore, $d(\mathscr{F}_1) = d(\mathscr{F}_k)$ for all k. This shows that a decision rule d defined by Eq. (4.18) satisfies the agreement condition for rings.

Conversely, suppose d satisfies the agreement condition for rings. Define the partial order $\leqq$ on $\{d(\mathscr{F}):\mathscr{F} \subset \mathscr{F}'\}$ as follows. We write $d(\mathscr{F}_1) \leqq d(F_2)$ if and only if there is an integer $n \geqq 1$ and σ-algebras $\mathscr{G}_k \subset \mathscr{F}'$, $1 \leqq k \leqq n$, such that $\sigma d(\mathscr{F}_1) \subset \mathscr{G}_1$, $\sigma d(\mathscr{G}_k) \subset \vee_{k+1}$ and $d(\mathscr{G}_n) = d(\mathscr{F}_2)$. It is easy to see that $d(\mathscr{F}) \leqq d(\mathscr{F})$ for all $\mathscr{F} \subset \mathscr{F}'$ (hence, $\leqq$ is reflexive), and that $d(\mathscr{F}_1) \leqq d(\mathscr{F}_2)$ and $d(\mathscr{F}_2) \leqq d(\mathscr{F}_3)$ imply $d(\mathscr{F}_1) \leqq d(\mathscr{F}_3)$ (hence, $\leqq$ is transitive). Suppose $d(\mathscr{F}_1) \leqq d(\mathscr{F}_2)$ and $d(\mathscr{F}_2) \leqq d(\mathscr{F}_1)$. Then there are σ-algebras $\mathscr{G}_k \subset \mathscr{F}'$, $1 \leqq k \leqq n+m$, such that $\sigma d(\mathscr{G}_{n+m}) \subset \mathscr{G}_1$, $\sigma d(\mathscr{G}_k) \subset \mathscr{G}_{k+1}$, $1 \leqq k \leqq n+m-1$, $d(\mathscr{F}_1) = d(\mathscr{G}_{n+m})$, and $d(\mathscr{F}_2) = d(\mathscr{G}_n)$. The agreement condition implies that $d(\mathscr{G}_k) = d(\mathscr{G}_1)$ for all k, and therefore, $d(\mathscr{F}_1) = d(\mathscr{F}_2)$. Thus, $\leqq$ is antisymmetric and hence, $\leqq$ is a partial order. Finally, if $\mathscr{G} \subset \mathscr{F}'$ and

$\sigma d(\mathscr{G}) \subset \mathscr{F}$, then $d(\mathscr{G}) \leqq d(\mathscr{F})$ by definition of $\leqq$. Thus, it follows that $d(F)$ is the maximum of $\{d(\mathscr{G}): \mathscr{G} \subset \mathscr{F}', \sigma d(\mathscr{G}) \subset \mathscr{F}\}$ with respect to $\leqq$.

In many cases, as in [12] and [13], the partial order relation is defined in terms of a scalar cost function. The following proposition proves that decision rules defined by such cost functions satisfy the agreement condition for rings provided that the decision includes a tie-breaking rule if the cost function has more than one minimum. This proposition shows, in particular, that the MAP decision rule defined in Section 2 satisfies the agreement condition for rings of decisionmakers.

PROPOSITION 4.4 *Suppose that decision functions take values in a set U. Let J be a real-valued functional of $\mathscr{F}'$-measurable decision functions $\delta: \Omega \to U$. For each $\mathscr{F}$ let $D(\mathscr{F})$ be the set of $\mathscr{F}$-measurable decision functions δ such that $J(\delta) \leqq J(\delta')$ for all $\mathscr{F}$-measurable $\delta': \Omega \to U$. Assume that U is partially ordered by $\leqq'$ and that for each $\mathscr{F}$, there is a $\delta \in D(\mathscr{F})$ such that $\delta' \in D(\mathscr{F})$ implies $\delta'(\omega) \leqq' \delta(\omega)$ for all $\omega \in \Omega$. The decision rule $d(\mathscr{F})$ which assigns this $\delta \in D(\mathscr{F})$ satisfies the agreement condition for rings.*

Proof Suppose $\delta_1, \delta_2: \Omega \to U$ are $\mathscr{F}'$-measurable. Define $\delta_1 \leqq'' \delta_2$ to mean that either $J(\delta_2) < J(\delta_1)$ or that $J(\delta_1) = J(\delta_2)$ and $\delta_1(\omega) \leqq' \delta_2(\omega)$ for all ω. It is easy to see that $\leqq''$ so defined partially orders all $\mathscr{F}'$-measurable decision functions. Suppose that $\mathscr{F} \subset \mathscr{F}'$ and δ is an $\mathscr{F}$-measurable decision function. Since $d(\mathscr{F}) \in D(\mathscr{F})$, by assumption $J(\delta) \leqq J(d(\mathscr{F}))$. If $J(\delta) = J(d(\mathscr{F}))$, then $\delta \in D(\mathscr{F})$ also, and $\delta(\omega) \leqq' d(\mathscr{F})(\omega)$ for all ω. It follows that $d(\mathscr{F})$ maximizes $\{\delta: \sigma\delta \in \mathscr{F}\}$ with respect to $\leqq''$. In particular, $d(\mathscr{F})$ maximizes $\{d(\mathscr{G}): \sigma d(\mathscr{G}) \subset \mathscr{F}, \mathscr{G} \subset \mathscr{F}'\}$. Thus, Proposition 4.3 implies that d satisfies the agreement condition for rings.

Typically, $J(\delta)$ will have the form

$$J(\delta) = E(c(\delta(\omega).\, \omega))$$

in which the expectation is over ω and $c: Ux\Omega \to R$ is a cost function. Additional assumptions will be necessary to ensure that J has $\mathscr{F}$-measurable minima (i.e., that $D(\mathscr{F}) \neq \varnothing$).

For example, suppose U is finite and define c as

$$c(u, \omega) = 1\{u = x(\omega)\}$$

where $x:\Omega\to U$ is a random U-valued function. Let $\leqq'$ be a total order of U (and therefore a partial order of U). Then the resulting decision rule is the MAP decision rule with the preference relation $\leqq'$. Consequently, we have the following.

COROLLARY 4.5 *The MAP decision rule of Section 2 satisfies the agreement condition for rings.*

Note that the fact that a decision rule is optimal with respect to some cost does not guarantee that the decision functions $d(\mathscr{F}_n)$ converge or that they converge to $\mathscr{F}=\bigvee_n \mathscr{F}_n$ when $\mathscr{F}_n \subset \mathscr{F}_{n+1}$. This is the case for the MAP decision rule. In this paper we have separated the common knowledge agreement condition from the continuity condition. The agreement condition is the essence of the agreement problem, even in the simplest cases. However, the continuity condition or something like it is necessary to deal with cases involving infinite measurement σ-algebras.

5. AGREEMENT AMONG DECISIONMAKERS WITH RANDOM COMMUNICATIONS

5.1 Introduction

Up to this point we have considered communication of random messages (i.e., decisions) among decisionmakers, but we have only allowed predetermined patterns of communication between decisionmakers. However, the results of [12] and [13] show that asymptotic agreement is also possible with random communication patterns that are not predetermined but depend on the course of events. The lattice dynamical system approach we have taken can easily treat predetermined communication patterns, but it is not obvious that it can also handle event-driven communication patterns.

In this section we show how to model random communication patterns in the lattice dynamical system framework introduced in Section 2. We then extend the results of Section 4 to the case of random communication patterns and thus obtain the results of [12] and [13] using our approach.

5.2 Event-dependent information algebras and random communication

Consider communications of the following type: if an event E occurs a message is received and the receiver knows this; if E does not occur, no message is received and the receiver knows this also. If $\mathscr{F}$ is a σ-algebra representing the message information, then the E-dependent message information σ-algebra can be described as follows. Define $\mathscr{F}/E$ to be the collection of all sets of the form $E \cap F$ or $(E \cap F) \cup E^c$ where $F \in \mathscr{F}$ and E^c is the complement of E with respect to the sample space Ω. Then $\mathscr{F}/E$ is a σ-algebra which represents the information in an E-dependent message when the receiver knows whether a message has arrived or not.

Note that $\mathscr{F}/\Omega = \mathscr{F}$. That is, the message information is completely received if the message is transmitted with certainty. Similarly, $\mathscr{F}/\varnothing = \{\varnothing, \Omega\}$. That is, there is no information if the message is never transmitted. Suppose E_n are events and $\bigcup_{n \geqq 1} E_n \supset E$. Then

$$\bigvee_{n \geqq 1} \mathscr{F}/E_n \supset \mathscr{F}/E \vee \mathscr{G}. \tag{5.1}$$

That is, if a message is transmitted repeatedly and at least one transmission is received whenever E occurs, then the message information is at least $\mathscr{F}/E$. Even if $\bigcup_n E_n = E$, there may not be equality in Eq. (5.1). Indeed, it is not hard to see that

$$\bigvee_{n \geqq 1} \mathscr{F}/E_n = \mathscr{F}/E \bigvee \mathscr{G} \tag{5.2}$$

where $\mathscr{G}$ is generated by the sets E_n.

Suppose that the information $\mathscr{F}(t)$ is available for transmission at time t, but the available information is only transmitted at random times τ^n. Suppose that the nth transmission is received at a random time σ^n. Using the $\mathscr{F}/E$ notation, we can represent the information received at time s by the expression

$$\bigvee_{t \geqq 0} \bigvee_{n \geqq 1} \mathscr{F}(t)/\{\tau^n = t \& \sigma^n = s\}.$$

Let us now formulate the lattice dynamical system equations for N decisionmakers who communicate at random times and whose messages are subject to random delays. Let τ_{jk}^n be the transmission time of the nth communication $(n \geqq 1)$ from decisionmaker j to decisionmaker k. Let σ_{jk}^n be the reception time of the nth communication $(n \geqq 1)$ from decisionmaker j to decisionmaker k. We allow the possibility that $\tau_{jk}^n = \infty$ or $\sigma_{jk}^n = \infty$ to indicate that no transmission was sent or none was received. We assume that $\tau_{jk}^n \leqq \tau_{jk}^{n+1}$ always and that $\tau_{jk}^n = \tau_{jk}^{n+1}$ only if $\tau_{jk}^n = \tau_{jk}^{n+1} = \infty$. The information σ-algebra received by k at time t is

$$\mathscr{R}_k(t) = \bigvee_{j \neq k} \bigvee_{s \geqq 0} \bigvee_{n \geqq 1} \sigma(d(\mathscr{F}_j(s)))/\{\tau_{jk}^n = s \& \sigma_{jk}^n = t\}. \tag{5.3}$$

The lattice dynamical system is given by Eq. (5.3) and

$$\mathscr{F}_k(t+1) = \mathscr{F}_k(t) \vee \mathscr{R}_k(t). \tag{5.4}$$

In the next subsection we investigate when the decisions $d(\mathscr{F}_k(t))$ all converge to the same decision for each k as $t \to \infty$.

5.3 Asymptotic agreement with random transmission times and random delays

To prove the following result about asymptotic agreement it is necessary to make certain technical assumptions. The argument we use follows the general outline of the proof of Propositions 3.1 and 4.2. There are two parts of the proof. The first part shows $\sigma d(\mathscr{F}_k) \subset \mathscr{F}_j$ for certain pairs k,j of decisionmakers and requires certain technical assumptions to handle convergence of decision functions. The second part deduces from the agreement condition for rings that the limiting decision functions all agree.

Recall from Section 3 that we assume that the underlying probability space is complete and that all σ-algebras we deal with contain the 0-probability events. Thus, we assume that the initial σ-algebras $\mathscr{Y}_k$ are complete with respect to the probability measure. Let us redefine $\sigma(d)$ and $\mathscr{F}/E$ to be the completion of these algebras as we originally defined them. Let us also say that j communicates infinitely often to k if $\tau_{jk}^n < \infty$ a.s. for each n and if $P(\{\sigma_{jk}^n < \infty$ for infinitely many $n\}) = 1$. This means that j transmits infinitely often to k and an infinite number of transmissions are actually received. With these technical assumptions and definitions we have the following proposition.

PROPOSITION 5.1 *Suppose that the decision rule d satisfies the agreement condition for rings and the continuity condition. Let A_k be* *the subset of decisionmakers who communicate infinitely often to k. Define $j \rightarrow k$, $j \leftrightarrow k$, and $[k]$ with respect to A_k just as in subsection 4.2. Then conclusions of Proposition 4.2 are true. That is, if $\{\mathscr{F}_k(t)\}_{t \geqq 0}$, $1 \geqq k \geqq N$, satisfy Eqs. (5.3) and (5.4) with initial conditions $\mathscr{F}_k(0) = \mathscr{Y}_k$, and if $\mathscr{F}_k = \bigvee_{t \geqq 0} \mathscr{F}_k(t)$, then*

$$\lim_{t \to \infty} d(\mathscr{F}_k(t)) = d(\mathscr{F}_k) \tag{5.5.}$$

and

$$d(\mathscr{F}_k) = d\left(\bigcap_{j \in [k]} \mathscr{F}_j\right). \tag{5.6}$$

for each k.

Proof Note that Eq. (5.5) follows from the continuity condition we assume that d satisfies. We will show that $\sigma d(\mathscr{F}_j) \subset \mathscr{F}_k$ whenever $j \in A_k$. The agreement condition for rings then implies Eq. (5.6) with the same argument as that of Section 4. Thus, we need only to prove $\sigma d(\mathscr{F}_j) \subset \mathscr{F}_k$ for $j \in A_k$.

Note that $\mathscr{R}_k(t) \subset \mathscr{F}_k(t+1)$ for each t and hence, $\mathscr{R}_k(t) \subset \mathscr{F}_k$ for each t. Thus, we have $\bigvee_{t \geqq 0} \mathscr{R}_k(t) \subset \mathscr{F}_k$. In particular, we see from Eq. (5.3) that for each $j \neq k$

$$\bigvee_{t \geqq 0} \bigvee_{s \geqq 0} \bigvee_{n \geqq 1} \sigma(d(\mathscr{F}_j(s)))/\{\tau^n_{jk} = s \& \sigma^n_{jk} = t\} \subset F_k. \tag{5.7}$$

Note that the order of the join operations $\bigvee$ in Eq. (5.7) is arbitrary, and Eq. (5.7) is equivalent to

$$\bigvee_{n \geqq 1} \bigvee_{s \geqq 0} \bigvee_{t \geqq 0} \sigma(d(\mathscr{F}_j(s)))/\{\tau^n_{jk} = s \& \sigma^n_{jk} = t\} \subset \mathscr{F}_k. \tag{5.8}$$

From the fact that

$$\bigcup_{t \geqq 0} \{\tau^n_{jk} = s \& \sigma^n_{jk} = t\} = \{\tau^n_{jk} = s \& \sigma^n_{jk} \neq \infty\}$$

and from Eq. (5.1) we can deduce from Eq. (5.8) that

$$\bigvee_{n \geqq 1} \bigvee_{s \geqq 0} \sigma(d(\mathscr{F}_j(s)))/\{\tau^n_{jk} = s \& \sigma^n_{jk} \neq \infty\} \subset \mathscr{F}_k. \tag{5.9}$$

Define the decision function d^n_{jk} as $d(\mathscr{F}_j(s))$ if $\tau^n_{jk} = s$. Since $\tau^n_{jk} < \infty$ a.s., d^n_{jk} is well-defined almost surely. The assumptions we made about completeness mean that the value of d^n_{jk} on sets of 0-probability does not matter. In particular, it does not affect $\sigma(d^n_{jk})$.

Note that

$$\sigma(d^n_{jk})/\{\tau^n_{jk} = s \& \sigma^n_{jk} \neq \infty\} = \sigma(d(\mathscr{F}_j(s)))/\{\tau^n_{jk} = s \& \sigma^n_{jk} \neq \infty\} \tag{5.10}$$

for each n, j, k, and s. From Eqs. (5.9) and (5.1), and the fact that

$$\bigcup_{s \geqq 0} \{\tau^n_{jk} = s \& \sigma^n_{jk} \neq \infty\} = \{\sigma^n_{jk} \neq \infty\} \tag{5.11}$$

we find that

$$\bigvee_{n \geqq 0} \sigma(d^n_{jk})/\{\sigma^n_{jk} \neq \infty\} \subset \mathscr{F}_k. \tag{5.12}$$

The following two lemmas will complete the proof of the proposition.

LEMMA 1

$$\lim_{n \to \infty} d^n_{jk} = d(\mathscr{F}_j).$$

LEMMA 2 *If* $\lim_{n \to \infty} f_n = f$ *and if* $P(\bigcap_{n \geqq 1} \bigcup_{m \geqq n} E_m) = 1$, *then*

$$\sigma(f) \bigvee_{n \geqq 1} \sigma(f_n)/E_n.$$

Completion of Proof of Proposition 5.1:
Note that if $E_n = \{\sigma^n_{jk} \neq \infty\}$, then

$$\{\sigma^n_{jk} \neq \infty \text{ for infinitely many } n\} = \bigcap_{n \geqq 1} \bigcup_{m \geqq n} E_m.$$

Thus, $P(\bigcap_{n \geqq 1} \bigcup_{m \geqq n} E_m) = 1$ if and only if $j \in A_k$. Consequently, $j \in A_k$ implies (by Lemmas 1 and 2) that $\sigma(d(\mathscr{F}_j)) \subset \mathscr{F}_k$, and the proposition is complete.

Proof of Lemma 1 Let us assume that the decision functions take values in a metric space with metric ρ. Egorov's theorem [21] asserts that there are measurable sets F_m such that $F_m \subset F_{m+1}$, $P(\cup_m F_m) = 1$, and

$$\rho(d(\mathscr{F}_j(t)), d(\mathscr{F}_j)) \leqq \varepsilon_m(t)$$

for $\omega \in F_m$, where $\varepsilon_m(t)$ is a real number independent of ω and $\varepsilon_m(t)$ decreases to 0 as $t \to \infty$. It follows that

$$\rho(d^n_{jk}, d(\mathscr{F}_j)) \leqq \varepsilon_m(t)$$

for $\omega \in F_m \cap \{\tau^N_{jk} \geqq t\}$ and $n \geqq N$. Thus,

$$\limsup_{n \to \infty} \rho(d^n_{jk}, d(\mathscr{F}_j)) \leqq \varepsilon_m(t) \tag{5.13}$$

for $\omega \in F_m \cap \{\tau^N_{jk} \geqq t\}$ for all N. By assumption, $G = \bigcup_N \{\tau^n_{jk} \geqq t\}$ has probability 1. We see that Eq. (5.13) is true for $\omega \in F_m \cap G$. Therefore,

$$\limsup_{n \to \infty} \rho(d^n_{jk}, d(\mathscr{F}_j)) = 0$$

for $\omega \in \bigcup_m F_m G$ and $P(\bigcup_m F_m \cap G) = 1$. This is the conclusion we wanted to prove.

Proof of Lemma 2 For $\omega \in \Omega$ define $v(1)$ (depending on ω) to be the smallest integer $n \geqq 1$ such that $\omega \in E_n$ or $v(1) = \infty$ if none exists. Define $v(2)$ to be the smallest integer $n > v(1)$ such that $\omega \in E_n$ or $v(2) = \infty$ if none exists. Define $v(l)$ similarly for all $l \geqq 1$. Note that $v(l) < \infty$ for all $l \geqq 1$ and $\lim v(l) = \infty$ for $\omega \in E = \bigcap_{nE1} \bigcup_{mEn} E_m$.

Define $g_l = f_{v(l)}$ for $\omega \in E$. Since $P(E) = 1$, g_l is well defined. Moreover, since $v(l) \leftrightarrow \infty$, it is clear that $g_l \to f$ as $l \to \infty$. Note that

$$\sigma(G_l)/\{v(l) = n\} = \sigma(f_n)/\{v(l) = n\}. \tag{5.14}$$

Moreover,

$$\bigcup_{l \geqq 1} \{v(l) = n\} = E_n \tag{5.15}$$

and

$$\bigcup_{n \geqq 1} \{\nu(l) = n\} = H_l. \tag{5.16}$$

Let $\mathscr{G}_n$ be the σ-algebra generated by the sets $\{\nu(l)=n\}$, $l \geqq 1$, and let $\mathscr{H}_l$ be the σ-algebra generated by the sets $\{\nu(l)=n\}$, $n \geqq 1$. Note that $\mathscr{F} = \bigvee_{n \geqq 1} \mathscr{G}_n = \bigvee_{l \geqq 1} \mathscr{H}_l$ is the σ-algebra generated by the sets E_n, $n \geqq 1$. Note that in Eq. (5.16) the sets H_l include E and hence $P(H_l) = 1$. Thus, $\mathscr{B}/H_l = \mathscr{B}$ for any complete σ-algebra $\mathscr{B}$.

Using Eq. (5.2) on Eq. (5.14) we see that

$$\bigvee_{l \geqq 1} \sigma(g_l) \vee \mathscr{F} = \bigvee_{n \geqq 1} \sigma(f_n)/E_n \vee \mathscr{F}.$$

Since $\mathscr{F} \subset \bigvee_{n \geqq 1} \sigma(f_n)/E_n$, we obtain

$$\bigvee_{l \geqq 1} \sigma(g_l) \subset \bigvee_{n \geqq 1} \sigma(f_n)/E_n.$$

Because $g_l \to f$, it follows that

$$\sigma(f \subset \bigvee_{l \geqq 1} \sigma(g_l)$$

and the proof is complete.

6. CONCLUSIONS

In this paper we have studied the problem of asymptotic agreement among communicating decisionmakers in terms of a lattice dynamical system defined on the lattice of n-tuples of information σ-algebras. This approach has allowed us to achieve a fundamental understanding of the problem of agreement among decisionmakers who communicate decisions based on general decision rules. We have shown that conditions necessary for agreement and continuity conditions necessary for convergence of decisions are clearly separate. We have also shown that the ring condition, necessary for agreement using indirect communication, is equivalent to having a decision rule which is optimal with respect to some preference relation between decision functions. Although we have focused our study on the asymptotic agreement problem, the lattice dynamical system formulation appears to be a useful approach to analyzing other problems of information evolution in distributed decisionmaking systems with specified communication and decision rules.

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