Correction to "An Efficient Game Form for Unicast Service Provisioning"

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Abstract—**A correction to the specification of the mechanism proposed in [1] is given.**

Index Terms— **Budget balance, game form/mechanism, individual rationality, Nash implementation, Unicast service provisioning.**

Due to an error, the mechanism presented in **[1]** has a tax function which is not differentiable with respect to the allocations. We need a tax function which is differentiable with respect to the allocations so that we can have Nash implementation. We correct this error as follows.

We consider the problem formulated in **[1]**. We use the same notation as in **[1]**.

Specification of the game form/mechanism:

Message space: The message space is the same as that of the mechanism presented in **[1]**. A message of user $i \in \mathcal{N}$ ($\mathcal{N}$ denotes the set of users) is of the form

$$m_i = (x_i, p_i^{l_{i_1}}, p_i^{l_{i_2}}, \cdots, p_i^{l_{i_{|\mathcal{R}_i|}}}),$$

where x_i denotes the (non-negative) bandwidth user i requests at all the links of his route, and $p_i^{l_{i_k}} \geq 0$ denotes the price user i is willing to pay per unit of bandwidth at link l_{j_k} of his route $\mathcal{R}_i$.

Outcome function: For any $m \in \mathcal{M}$, the outcome function is defined as follows:

$$f(m) = (x_1, x_2, \cdots, x_n, t_1, t_2, \cdots, t_n)$$
$$t_i = \sum_{l \in \mathcal{R}_i} t_i^l,$$

where t_i^l is the tax paid by user i for using link l. The form of t_i^l is the same as the tax function defined in **[1]** excluding the term that is of the form described by relation **(23) in [1]**. For example, if $|\mathcal{G}^l| > 3$, ($\mathcal{G}^l$ denotes the set of users using link l) the tax function in Eq. **(13) of [1]** now becomes,

$$t_i^l = P_{-i}^l x_i + (p_i^l - P_{-i}^l - \zeta_+^l)^2 - 2P_{-i}^l \left(p_i^l - P_{-i}^l\right) \left(\frac{\mathcal{E}_{-i}^l + x_i}{\gamma}\right) + \Phi_i^l, \qquad (1)$$

where

$$\zeta_+^l = \max\{0, \frac{\sum_{i \in \mathcal{G}^l} x_i - c^l}{\hat{\gamma}}\}, \qquad (2)$$

c^l is the capacity of link l, Φ_i^l is defined by Eq. **(14) in [1]**,

$$P_{-i}^l = \frac{\sum_{\substack{j \in \mathcal{G}^l \\ j \neq i}} p_j^l}{|\mathcal{G}^l| - 1}, \qquad \mathcal{E}_{-i}^l = \sum_{\substack{j \in \mathcal{G}^l \\ j \neq i}} x_j - c^l, \qquad (3)$$

(P_{-i}^l and $\mathcal{E}_{-i}^l$ are the same as in **[1]**) and $\gamma, \hat{\gamma}$, are positive constants.

This completes the specification of the mechanism.

Based on the above specification, the proof of Lemma **2 in [1]** is updated as follows.

Proof of Lemma 2 in [1]: Let $m^* = (m_i^*, m_{-i}^*)$ be a NE of the game induced by the mechanism. Since user i does not control Φ_i^l, it implies $\frac{\partial \Phi_i^l}{\partial p_i^l} = 0$, (as in Eq. **(34) of [1]**). By following the same steps as in equations **(35-38) of [1]**, we obtain for any $l \in \mathbf{L}$:

$$\frac{\partial t_i^l}{\partial p_i^l}\Big|_{m=m^*} = 2\left[(p_i^{*l} - P_{-i}^{*l} - \zeta_+^{*l}) - P_{-i}\left(\frac{\mathcal{E}_{-i}^{*l} + x_i^*}{\gamma}\right)\right] = 0. \qquad (4)$$

Summing (4) over all $i \in \mathcal{G}^l$, we get

$$\begin{aligned}\sum_{i \in \mathcal{G}^l} \frac{\partial t_i^l}{\partial p_i^l}\Big|_{m=m^*} &= \sum_{i \in \mathcal{G}^l}\left[(p_i^{*l} - P_{-i}^{*l} - \zeta_+^{*l}) - P_{-i}\left(\frac{\mathcal{E}_{-i}^{*l} + x_i^*}{\gamma}\right)\right] \\ &= -|\mathcal{G}^l|\zeta_+^{*l} - \sum_{i \in \mathcal{G}^l} P_{-i}^{*l}\left(\frac{\mathcal{E}_{-i}^{*l} + x_i^*}{\gamma}\right) \\ &= 0. \end{aligned} \qquad (5)$$

Suppose $\sum_{i \in \mathcal{G}^l} x_i^* > c^l$. Then we must have, $\zeta_+^{*l} > 0$ and $\sum_{i \in \mathcal{G}^l} P_{-i}^{*l}\left(\frac{\mathcal{E}_{-i}^{*l} + x_i^*}{\gamma}\right) \geq 0$. But this contradicts Eq. (5). Therefore, we must have

$$\sum_{i \in \mathcal{G}^l} x_i^* \leq c^l. \qquad (6)$$

This implies,

$$\zeta_+^{*l} = 0. \qquad (7)$$

Combining (7) along with (5) we obtain

$$\sum_{i \in \mathcal{G}^l} P_{-i}^{*l}\left(\frac{\mathcal{E}_{-i}^{*l} + x_i^*}{\gamma}\right) = 0. \qquad (8)$$

Moreover, combining (6) and (8) we obtain

$$P_{-i}^{*l}\left(\frac{\mathcal{E}_{-i}^{*l} + x_i^*}{\gamma}\right) = 0. \qquad (9)$$

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for every $i \in \mathcal{G}^l$. Using (7) and (9) in (4) we obtain

$$p_i^{*l} = P_{-i}^{*l}. \tag{10}$$

Since (10) is true for all $i \in \mathcal{G}^l$, it implies,

$$p_i^{*l} = p_j^{*l} = P_{-i}^{*l} =: p^{*l}, \tag{11}$$

and along with (9) it implies

$$p^{*l}\mathcal{E}^{*l} = 0, \tag{12}$$

where $\mathcal{E}^{*l} = \sum_{i \in \mathcal{G}^l} x_i^* - c^l$ ($\mathcal{E}^{*l}$ is the same as in **[1]**).

Furthermore, since

$$\frac{\partial \Phi_i^l}{\partial x_i} = 0 \tag{13}$$

(Eq. **(34)** in **[1]**)), it follows from (1) that

$$\frac{\partial t_i^l}{\partial x_i}\Big|_{m=m^*} = p^{*l}. \tag{14}$$

because of (7), (11), (12), and (13). ■

Remark 1. The proof of Theorem 5 follows when $x_i^* > 0$. Note that, when $x_i^* = 0$, since user i does not have incentive to increase its demand, it follows that

$$\frac{\partial \mathbf{U}_i(x_i)}{\partial x_i} - \sum_{l \in \mathcal{R}_i} p^{*l}\Big|_{m=m^*} \leq 0. \tag{15}$$

Now, set $\lambda^{*l} = p^{*l}$. Then (12) and (15) are consistent with the KKT conditions **(68-70) of [1]**.

I. Properties of the mechanism

Existence of Nash equilibria (NE): The proof of existence of NE of the game induced by the mechanism is the same as in **[1]** (see Theorem **6**, page **398**, and its proof in **[1]**; also see the proof of Theorem **7**).

Feasibility of allocations at NE: Because of the specification of the mechanism and Eq. (7), the allocations corresponding to all NE are in the feasible set.

Budget Balance at any feasible allocation: Budget balance at any feasible allocation follows by Lemma **3** of **[1]**.

Individual Rationality: Individual rationality follows by Theorem **4** of **[1]**.

Nash implementation: Nash implementation follows by Theorem **5** of **[1]**.

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References

[1] A. Kakhbod, D. Teneketzis, *An efficient game form for unicast service provisioning*. IEEE Transactions on Automatic Control, Vol 57, No. 2, February 2012, pp. 392-404.