JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 54, No. 2, AUGUST 1987

TECHNICAL NOTE

Informational Aspects of a Class of Subjective Games of Incomplete Information: Static Case[1]

D. TENEKETZIS[2] AND D. A. CASTANON[3]

Communicated by Y. C. Ho

Abstract. Subjective games of incomplete information are formulated where some of the key assumptions of Bayesian games of incomplete information are relaxed. The issues arising because of the new formulation are studied in the context of a class of nonzero-sum, two-person games, where each player has a different model of the game. The static game is investigated in this note. It is shown that the properties of the static subjective game are different from those of the corresponding Bayesian game. Counterintuitive outcomes of the game can occur because of the different beliefs of the players. These outcomes may lead the players to realize the differences in their models.

Key Words. Subjective games, incomplete information, public information, private information, secret information, value of information.

1. Introduction

Game theory is the mathematical science which studies decision making in situations of potential conflict among decisionmakers. The requirements of formal game theory are strict regarding the rules of the game and the portrayal of exogenous uncertainty. Due to these requirements, there are many strategic situations which cannot be initially modeled as games because players lack information about available strategies, utility functions, or outcomes resulting from various strategies.

Specifically, the key requirements of formal game theory are:

(A1) the rules of the game are common information (Ref. 1) to all players of the game;

[1] This work was sponsored by the Office of Naval Research under Contract No. N00014-84-C-0485.

[2] Associate Professor, Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, Michigan.

[3] Senior Research Scientist, ALPHATECH, Burlington, Massachusetts.

0022-3239/87/0800-0413$05.00/0 © 1987 Plenum Publishing Corporation

(A2) exogenous uncertainty is portrayed by objective probabilities which are common knowledge (Ref. 1) to all players;

(A3) players are fully committed to *a priori* strategies;

(A4) players are rational.

As game theory developed, attempts were made to relax some of these assumptions. Requirement (A3) was a consequence of the normalization principle of Von Neumann, (Ref. 2); Aumann and Maschler (Ref. 3) were the first to point out, via a simple counterexample, the inappropriateness of the normalization principle under certain conditions; since then, considerable developments followed by relaxing the requirement of prior commitment (Refs. 4–8).

Harsanyi (Ref. 9) and Aumann-Maschler *et al.* (Ref. 10) pointed out that, in some military problems, players may lack full information about the payoff functions of other players, or about the physical facilities and strategies of other players, or even about the amount of information that other players have about the various aspects of the game situation. Thus, Harsanyi (Ref. 9) first relaxed requirement (A1) and formulated and developed models of games of incomplete information. Harsanyi modeled the incomplete information as an exogenous random move (nature's move), selecting one of a finite number of possible games; he also assumed that the outcomes of this move have a subjective probability distribution which is common knowledge to all players. Considerable progress has been achieved in the theory of games of incomplete information using Harsanyi's original formulation (see Refs. 10–12 and references therein.)

A restriction in Harsanyi's formulation is the requirement of common knowledge of the probability distribution of nature's move. In many strategic situations (especially in noncooperative games), this distribution is subjectively assessed by each player, and subject to individual biases and inaccuracies (Ref. 13). In this paper, we formulate a class of games, which we call subjective games of incomplete information, which relaxes Harsanyi's requirements of common knowledge. Specifically, we allow each person to have his own subjective probability distribution of nature's move; in addition, each person believes his subjective distribution is common information, whereas it is actually secret information (Ref. 14). As a consequence, requirements (A1) and (A2) are relaxed, and requirement (A4) is modified, in the sense that each player is considered to be rational within his/her own subjective view of the game. Various interesting issues arise because of our formulation:

(Q1) How are equilibrium strategies defined for subjective games?

(Q2) How do these equilibrium strategies relate to the equilibrium strategies of the games studied so far?

(Q3) Does repetition of the game result in cooperation as in the case of the games studied so far (e.g., Ref. 15)? Does repetition of the game alleviate differences in the subjective assessments of the players and allow players to agree on an equilibrium strategy? Is it possible to characterize the set of all equilibria for repeated subjective games?

To understand some of the questions above, we shall consider a special class of games, namely, 2×2 two-person, nonzero sum games of incomplete information where the payoff matrices have a special structure. In this note, we shall consider the static game and, in a subsequent note (Ref. 16), the infinitely repeated game. The rest of this paper is organized as follows. In Section 2, we present the model for subjective games, and discuss briefly games of incomplete information and point out the differences between Harsanyi's model and our model. In Section 3, we study a static subjective noncooperative game of incomplete information. We consider the problem under four different types of information that a player may receive: (i) no information; (ii) public information; (iii) private information; (iv) secret information. For each case, we investigate the players' rational strategies and we define and determine the value of information. We also investigate the conditions under which the players realize the differences in their models. Conclusions are presented in Section 4.

2. Formulation of Subjective Games of Incomplete Information

We shall develop our theory of subjective games based on the following key assumptions:

(S1) players have different probability assessments on nature's move;
(S2) each player thinks that the other players' assessments are the same as his;
(S3) players are Bayesian;
(S4) each player is rational within his own subjective view of the game.

Assumption (S2) implies that the rules of the game are not common knowledge to all the players, since each player thinks that the other players' assessments are the same as his, yet this may not be true. Assumptions (S1) and (S2) were previously used in the context of distributed estimation and detection (Ref. 17).

More precisely, let b_i represent the private information of player i about the game. This information relates to the outcome of nature's move. In dealing with incomplete information, each player takes a Bayesian approach. That is, each player assigns a subjective probability distribution

$$P_i = P_i(b_1, b_2, \ldots, b_i, \ldots, b_n)$$

to nature's move and attempts to maximize the mathematical expectation of his own payoff J_i in terms of this probability distribution. Furthermore, each player i assumes that $P_i = P_j$, for all j, whereas in the actual game P_i and P_j may be different.

Comparing the mathematical model described above with Harsanyi's formulation, we note that Harsanyi also assumes that each player assigns a subjective probability distribution P_i to nature's move; although P_i and P_j may differ, all the distributions P_i are assumed to be common knowledge to all players. In our formulation, any difference in subjective probabilities is secret information. Moreover, each player is unaware that he has secret information.

3. Static Subjective Noncooperative Games of Incomplete Information

Problem Formulation. We consider the following static two-person, nonzero sum game. Nature selects one of two games with the following payoff matrices:

$$\text{game 1:}\quad \begin{array}{c} \\ \lambda \\ \mu \end{array}\!\!\begin{array}{c} \quad\sigma \qquad\quad \tau \\ \begin{bmatrix} (a, a) & (c, b) \\ (b, c) & (d, d) \end{bmatrix} \end{array}; \tag{1}$$

$$\text{game 2:}\quad \begin{array}{c} \\ \lambda \\ \mu \end{array}\!\!\begin{array}{c} \quad\sigma \qquad\quad \tau \\ \begin{bmatrix} (b, b) & (d, a) \\ (a, d) & (c, c) \end{bmatrix} \end{array}. \tag{2}$$

We further assume that

$$a > c > b > d, \tag{3}$$

$$b + c > a + d. \tag{4}$$

Player 1 can choose action λ or μ and player 2 can choose action σ or τ. Note that, because of (3), each player has a dominant strategy in each one of the two games. So far, the statement of the problem and the assumptions (3)–(4) are essentially the same as in Ref. 14. However, contrary to Ref. 14, we now assume that the two players have a different probability assessment of nature's move. Let r be the true probability that nature selects game 1. Let p, q be player 1's and player 2's assessments of this event, respectively. Assume that

$$p > 1/2, \qquad q < 1/2.$$

We will consider this problem under four different types of information that a player may receive:

(C1) no information: in this case, none of the players is informed about the outcome of nature's move;

(C2) public information: in this case, both players are informed about the outcome of nature's move;

(C3) private information; in this case, one player is informed about the outcome of nature's move, whereas the other player is not; moreover, this distribution of information is common knowledge;

(C4) secret information: in this case, one player is informed about the outcome of nature's move whereas the other player is uninformed; moreover, the uninformed player is unaware that his opponent is informed, and the informed player knows this.

The rational strategies in each of these situations are given below.

(C1) No Information. In this case, player 1 plays λ and player 2 plays τ. The payoffs of the two players are

$$J^{0_1} = rc + (1-r)d, \tag{5}$$

$$J^{0_2} = rb + (1-r)a. \tag{6}$$

(C2) Public Information. In this case, player 1 plays λ in game 1 and μ in game 2. Player 2 plays σ in game 1 and τ in game 2. Thus, the payoff of the players is

$$J^{B_1} = J^{B_2} = ra + (1-r)c. \tag{7}$$

Define the value of information as follows (see also Ref. 18): V_i, the value of information to player i, is the payoff of player i when he knows the outcome of nature's move minus the payoff of player i when no player is informed about the outcome of nature's move. In this case, the value of public information for players 1 and 2 is given by

$$V^{B_1} = r(a-c) + (1-r)(c-d), \tag{8}$$

$$V^{B_2} = (2r-1)a + (1-r)c - rb. \tag{9}$$

(C3) Private Information. Assume at first that player 1 is the informed player. Then he plays λ in game 1 and μ in game 2. Player 2 plays τ. The payoffs of the two players are

$$J^{P_1} = c, \tag{10}$$

$$J^{P_2} = rb + (1-r)c. \tag{11}$$

If player 2 is the informed player, then he plays σ in game 1 and τ in game 2. If

$$p < (c-d)/(a+c-b-d) = p^*, \tag{12}$$

then player 1 plays μ. Otherwise, he plays λ. The expected payoffs for player 1 are then

$$J^{P_1} = rb + (1-r)c, \tag{13}$$

$$J^{P_1} = ra + (1-r)d, \tag{14}$$

respectively. The payoff for player 2 is

$$J^{P_2} = c(\text{corresponding to } \mu), \tag{15}$$

$$J^{P_2} = a(\text{corresponding to } \lambda), \tag{16}$$

The value of information for the two players is

$$V^{P_1} = (1-r)(c-d), \tag{17}$$

$$V^{P_2} = c - rb - (1-r)a, \qquad \text{if } p < p^*, \tag{18a}$$

$$V^{P_2} = r(a-b), \qquad \text{otherwise.} \tag{18b}$$

(C4) Secret Information. Assume at first that player 1 is informed secretly about the outcome of nature's move. Then, he plays λ in game 1 and μ in game 2. Player 2 plays τ. The payoffs of the two players are

$$J^{C_1} = c, \tag{19}$$

$$J^{C_2} = rb + (1-r)c. \tag{20}$$

The value of secret information to player 1 in this case is

$$V^{S_1} = (1-r)(c-d). \tag{21}$$

Assume now that player 2 is secretly informed. Then, he plays σ in game 1 and τ in game 2. Player 1 plays λ. The payoffs of the two players are

$$J^{C_1} = ra + (1-r)d, \tag{22}$$

$$J^{C_2} = a. \tag{23}$$

The value of secret information to player 2 in this case is

$$V^{S_2} = r(a-b). \tag{24}$$

Let us discuss some interesting features of the solutions of these games. At first, note that each payoff bimatrix is symmetric; hence, in each one of the two games, the players are interchangeable. Thus, one expects that, for the classical Bayesian game, in the case of public or secret information, the behavior of the informed and the uninformed player will be independent of who is the informed and who is the uninformed player. For example, in the case of private or secret information, if player 1 were the uninformed player and played λ, we would expect that, if the situation were reversed and player 2 became the uninformed player, he would play σ. Also, in the case where no player was informed about the outcome of the chance move the dominant strategies would be (λ, σ) or (μ, τ). Consequently, the value of private, secret, or public information would be the same for both players. It can be checked easily that this is indeed the case when $p = q = r$. However, this behavior is not observed when each player has his own subjective model of the game. When player 1 is privately informed about nature's move, player 2 always chooses τ (the second column); on the other hand, if player 2 is informed privately about the outcome of nature's move, player 1 does not always play μ (the second row). When player 1 is the secretly informed player, player 2 always plays τ (second column); if player 2 is the secretly informed player, player 1 always plays λ (first row). When no player is informed about the outcome of the chance move, the outcome of the game is (λ, τ). These facts indicate that the value of private and secret information is now different for each player, as is evident from the analysis above.

For the class of games considered in this section, the value of public, private, and secret information differs from player to player, whereas, in the classical Bayesian framework, this value does not depend on who is the informed and who is the uninformed player. This phenomenon is due to the differences in the initial probability assessments of the incomplete information.

Another interesting observation follows from the previous results. Consider the case where player 2 is privately informed, $p < p^*$, and $r = 1/2$. Then, the value of information for player 2 is given by

$$V_2^P = c - 0.5b - 0.5a.$$

If

$$c < 0.5a + 0.5b,$$

the value of private information for player 2 is negative! On the other hand, the gain for player 1, the uninformed player, is equal to $0.5\,(b - d)$, which is positive. Thus, for the class of symmetric games considered in this paper, we have a case where the value of private information is negative for the informed player and the uninformed player benefits from the situation! This phenomenon never occurs for this class of games in the classical Bayesian framework, where, if the value of private information is negative for the informed player, the uninformed player cannot benefit either (Ref. 13). Even more surprising in this case is the fact that the informed player wants to use his private information, whereas the uninformed player wishes that the informed player acted as if he were not informed!

The reason for all these counterintuitive results and the differences between the subjective game results and the classical Bayesian game results is that each player evaluates the game as well as the behavior of his opponent in the game in terms of his own model and acts accordingly. Such subjective evaluations lead to behavior which would never occur in the classical Bayesian formulation as evidenced by the previous analysis.

One issue that arises naturally in these games is the following: How do the players involved in the game interpret its outcome? Do they realize that they have different models? If neither player is informed about the outcome of nature's move, player 1 expects that player 2 will use strategy σ and player 2 expects that player 1 will use strategy μ. At the end of the game, each player finds out that the outcome is the opposite of what he expected. Since each player assumes that his opponent is rational, both players conclude that they have different models. Similar phenomena occur if one of the players is either secretly or privately informed.

In the case of secret information, the secretly informed player discovers at the end of the game that his opponent's perception of the game is different from his. On the other hand, the uninformed player may never discover that his opponent has a different perception of the game, or he may not be able to interpret his opponent's move in terms of his own model, in which case he can conclude that either his opponent has a different model of the game, or his opponent has secret information.

In the case of private information, the uninformed player is not in a position to discover at the end of the game that his opponent has a different view of the game. The informed player may or may not discover at the end of the game that he and his opponent have inconsistent beliefs about the game, depending on whether Eq. (12) holds.

Note that, if both $p, q > 1/2$ or $p, q < 1/2$, the players never discover the differences in their model.

4. Conclusions

In this note, we formulated a class of *subjective games*, where the players have different perceptions of the rules of the game and are unaware of the differences in their perception. We studied in detail a specific class of static symmetric games of incomplete information, and we showed that the properties of these subjective games are different from the properties of similar Bayesian games. Specifically, many features of the Bayesian games, such as the positive value of private informtion in symmetric games, are not maintained when the players' perceptions of the game are allowed to differ. The inconsistent beliefs of the players lead to counterintuitive behavior.

The rudimentary investigation reported here needs to be carried further. An important issue which has not been addressed so far is whether the differences in beliefs between the two players are amplified or smoothed out if the game is repeated infinitely many times. In addition, our analysis should be extended beyond the point where the two players reach an impasse. Some of these issues, namely, the effect of the infinite repetition of the game as well as the effect of the bargaining models adopted by the players on the outcome of the game, will be addressed in a forthcoming paper (Ref. 16).

References

1. WASHBURN, R. B., and TENEKETZIS, D., *Asymptotic Agreement among Communicating Decisionmakers*, Stochastics, Vol. 13, pp. 103-129, 1984.
2. VON NEUMANN, J., and MORGENSTERN, O., *Theory of Games and Economic Behavior*, Wiley, New York, New York, 1944.
3. AUMANN, R. J., and MASCHLER, M., *Some Thoughts on the Minimax Principle*, Management Science, Vol. 18, pp. 54-63, 1972.
4. HO, Y. C., *On the Minimax Principle and Zero-Sum Stochastic Differential Games*, Harvard University, Cambridge, Massachusetts, Technical Report No. 630, 1972.
5. KREPS, D. M., and WILSON, R., *Sequential Equilibria*, Stanford University, Stanford, California, Preprint, 1980.
6. WILSON, R., *Perfect Equilibria and Sequential Rationality*, Stanford University, Stanford, California, Preprint, 1981.
7. SELTEN, R., *Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games*, International Journal of Game Theory, Vol. 4, pp. 25-55, 1975.
8. SELTEN, R., *Spieltheoretische Behandlung eines Oligopolmodels mit Nachfragetragheit*, Zeitschrift fur die Gesamte Statswissenschaft, Vol. 12, pp. 301-324, 1965.
9. HARSANYI, J. C., *Games of Incomplete Information Played by Bayesian Players, Parts I-III*, Management Science, Vol. 14, pp. 159-182, 1967; Vol. 14, pp. 320-334, 1967; and Vol. 14, pp. 486-502, 1967.
10. AUMANN, R. J., MASCHLER, M., HARSANYI, J. C., MAYBERRY, J. P., SCARF, H. E., SELTEN, R., and STEARNS, R. *Game Theoretic Aspects of Gradual Disarmament*, Mathematical Report to Arms Control and Disarmament Agency, 1966-1968.
11. SORIN, S., *An Introduction to Two-Person Zero-Sum Repeated Games of Incomplete Information*, Stanford University, Institute for Mathematical Studies in the Social Sciences, Technical Report No. 312, 1980.
12. HART, S., *Nonzero Sum Two-Person Repeated Games with Incomplete Information*, Stanford University, Institute for Mathematical Studies in the Social Sciences, Technical Report No. 367, 1982.

13. TVERSKY, A., and KAHNEMAN, D., *Judgment under Uncertainty: Heuristics and Biases*, Science, Vol. 185, pp. 1124-1131, 1974.
14. LEVINE, P., and PONSSARD, J. P., *The Values of Information in Some Nonzero Sum Games*, International Journal of Game Theory, Vol. 6, pp. 221-229, 1977.
15. AUMANN, R. J., *Acceptable Points in General Cooperative n-Person Games*, Contributions to the Theory of Games, Vol. IV, Annals of Mathematical Studies, No. 40, pp. 287-324, Princeton University Press, Princeton, New Jersey, 1959.
16. TENEKETZIS, D., and CASTANON, D. A., *Informational Aspects of Subjective Games of Incomplete Information: The Infinitely Repeated Case* (to appear).
17. TENEKETZIS, D., and VARAIYA, P., *Consensus in Distributed Estimation with Inconsistent Beliefs*, Systems and Control Letters, Vol. 4, pp. 217-221, 1984.
18. SUN, F. K., and HO, Y. C., *Value of Information in Zero-Sum Games*, Mathematical Programming Study, Vol. 5, pp. 211-226, 1976.