On the Design of Globally Optimal Communication Strategies for Real-Time Noisy Communication Systems with Noisy Feedback

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***Abstract*—A real-time communication system with noisy feedback is considered. The system consists of a Markov source, forward and backward discrete memoryless channels, and a receiver with limited memory. The receiver can send messages to the encoder over the backward noisy channel. The encoding at the encoder and the decoding, the feedback, and the memory update at the receiver must be done in real-time. A distortion metric that does not tolerate delays is given. The objective is to design an optimal real-time communication strategy, i.e., design optimal real-time encoding, decoding, feedback, and memory update strategies to minimize a total expected distortion over a finite horizon. This problem is formulated as a decentralized stochastic optimization problem and a methodology for its sequential decomposition is presented. This results in a set of nested optimality equations that can be used to sequentially determine optimal communication strategies. The methodology exponentially simplifies the search for determining an optimal real-time communication strategy.**

***Index Terms*—Markov decision processes, real-time communication, noisy feedback, dynamic teams, information state, common knowledge, common belief**

I. Introduction

THE CLASSICAL Shannon's formulation of a communication system [1] does not take communication delay into account. For many applications such as sensor networks, transportation networks, and networked controlled systems, in which the communication system is a component of a larger system, the delay incurred during the transmission of information has to be bounded. This motivates the need to study communication systems with a hard constraint on communication delay. Such communication systems are called *real-time communication systems*.

Real-time communication problems are drastically different from classical information theoretic formulations. The fundamental concepts of information theory such as source entropy, rate distortion, and channel capacity are asymptotic concepts and do not provide much insight for the real-time communication problem. Due to the real-time constraint on information transmission, the separation of source and channel coding is no longer optimal. Therefore, in real-time communication systems, joint source-channel coding schemes must be considered.

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Real-time communication problems can be viewed as multi-agent sequential stochastic optimization problems with decentralized information. The system has two agents (or decision makers) — the encoder and the decoder. Due to noise in the communication channel, one agent does not know the information available at the other agent; so, the information in the system is decentralized. Consequently, the resultant stochastic optimization problem cannot be solved using Markov decision theory [2] since Markov decision theory is only appropriate for problems with centralized information.

Several variations of the real-time communication problem have been considered in the literature. The research on real-time communication can be broadly classified into three categories: (i) performance bounds of finite-delay or real-time communication systems; (ii) real-time encoding and decoding of individual sequences; and (iii) real-time encoding and decoding of Markov sources.

Performance bounds for real-time (zero-delay or finite-delay) communication systems under various assumptions were derived in [3]–[11]. The design of asymptotically optimal real-time encoding and decoding strategies for noiseless channels was considered in [12]–[14], and for noisy channels was considered in [15]. Structural properties of optimal real-time encoders for transmitting Markov sources over noiseless channel were derived in [16]–[18]. Structural properties of optimal real-time encoders and decoders for transmitting Markov sources over noisy channels with noiseless feedback was considered in [19]–[22]. Structural properties of optimal real-time encoders and decoders for transmitting Markov sources over noisy channels was considered in [23]; a methodology for determining globally optimal encoding and decoding strategies for communication systems with noisy channels and no feedback was presented in [24], [25].

The work on real-time communication, summarized above, either assumes a noiseless channel, or a noisy channel with no feedback, or a noisy channel with noiseless feedback. The design of optimal real-time communication strategies for systems with noisy channels and noisy feedback has not been considered so far. The problem of communicating over a noisy channel with noisy feedback has been considered in the information theoretic setting [26]–[29], but these results do not assume a real-time constraint on information transmission.

The key contributions of this paper are: (i) the presentation of a systematic methodology for the design of globally optimal strategies for real-time communication systems with noisy feedback; and (ii) an explaination of this solution methodology. We treat the design of an optimal communication strategy as a decentralized multi-agent sequential optimization problem. We present a methodology that allows us to sequentially determine an optimal communication strategy by proceeding backward in time and solving a set of nested optimality equations. This methodology drastically simplifies the search for an optimal real-time communication strategy; in spite of this simplification, numerically solving the resultant nested optimality equations remains a formidable task.

The rest of this paper is organized as follows. In Section II we formulate the problem and present its salient features. In Section III we present the structural/qualitative properties of optimal encoders and decoders. In Section IV we compare the structural properties derived in this paper with the previously known structural properties for real-time communication systems. In Section V we present the methodology for sequentially determining globally optimal communication strategies. In Section VI we discuss different approaches to decentralized optimization problems, and explain why the proposed solution methodology works. In Section VII we mention some possible extensions and we conclude in Section VIII.

Notation: Throughout this paper we use the following notion. Uppercase letters (X, Y, Z) represent random variables, lowercase letters (x, y, z) represent their realizations, and calligraphic letters $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ represent their alphabets. Script letters $(\mathscr{C}, \mathscr{G}, \mathscr{L})$ represent family of functions and Gothic letters $(\mathfrak{F}, \mathfrak{E}, \mathfrak{R})$ represent σ-algebras. For random variables and functions, x^t is a short hand for the sequence $x_1, \dots, x_t$. $\mathbb{E}\{\cdot\}$ denotes the expectation of a random variable, $\Pr(\cdot)$ denotes the probability of an event, $\mathbb{I}[\cdot]$ denotes the indicator function of a statement, and $\mathbb{P}\{\mathcal{X}\}$ denotes the space of all PMF (probability mass functions) on $\mathcal{X}$. To denote that the expectation of a random variable or the probability of an event depends on a function φ, we use $\mathbb{E}\{\cdot \mid \varphi\}$ and $\Pr(\cdot \mid \varphi)$, respectively. This slightly unusual notation is chosen since we want to keep track of all functional dependencies and the conventional notation of $\mathbb{E}^{\varphi}\{\cdot\}$ and $\Pr^{\varphi}(\cdot)$ is too cumbersome to use. $\mathbb{B}(\mathcal{X})$ denotes the Borel σ-field on $\mathcal{X}$. If P is a probability measure on a σ-field $\mathfrak{F}$, and $\mathfrak{G}$ is a subfield of $\mathfrak{F}$, then $P\big|_{\mathfrak{G}}$ denotes the restriction on P onto $\mathfrak{G}$.

II. The Finite Horizon Problem

A. Problem Formulation

Consider a real-time communication system with noisy feedback shown in Figure 1. This system consists of a source, a real-time encoder, a noisy forward channel, a noisy backward channel, and a real-time decoder with finite memory. The communication system operates in discrete time for a time horizon T.

At each stage t, the source produces an output X_t taking values in a finite alphabet $\mathcal{X}$. We assume that the output sequence $\{X_t,\ t = 1, \dots, T\}$ forms a first-order Markov chain with initial distribution P_{X_1} and matrix of transition probabilities $P_{X_{t+1}|X_t}$.

The communication system consists of two channels: the forward channel and the backward channel. We assume that both channels are independent DMC (discrete memoryless channels). The forward channel is a $|\mathcal{Z}|$-input $|\mathcal{Y}|$-output DMC, while the backward channel is a $|\tilde{\mathcal{Z}}|$-input $|\tilde{\mathcal{Y}}|$-output DMC. These channels can be described as

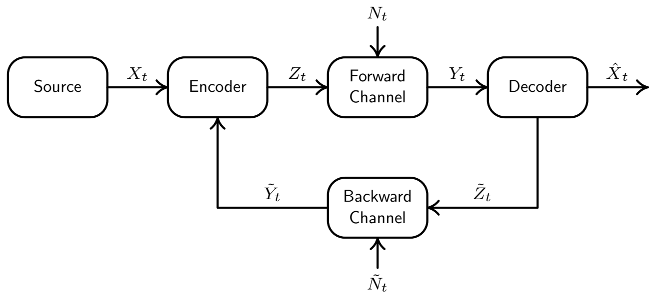

Fig. 1. A real-time communication system with noisy feedback

$$Y_t = h(Z_t, N_t), \qquad t = 1, \dots, T, \tag{1a}$$
$$\tilde{Y}_{t-1} = \tilde{h}(\tilde{Z}_{t-1}, \tilde{N}_{t-1}), \qquad t = 2, \dots, T, \tag{1b}$$

where $h(\cdot)$ and $\tilde{h}(\cdot)$ denote the forward and backward channels at time t, respectively; Z_t and $\tilde{Z}_{t-1}$ are the inputs to the forward and the backward channels at time t, respectively; Y_t and $\tilde{Y}_{t-1}$ are the outputs of the forward and the backward channels at time t, respectively; and N_t and $\tilde{N}_{t-1}$ are the channel noise in the forward and the backward channels at time t, respectively. The variables Z_t, $\tilde{Z}_t$, Y_t, $\tilde{Y}_t$, N_t, and $\tilde{N}_t$ take values in finite alphabets $\mathcal{Z}$, $\tilde{\mathcal{Z}}$, $\mathcal{Y}$, $\tilde{\mathcal{Y}}$, $\mathcal{N}$, and $\tilde{\mathcal{N}}$, respectively. We assume that $\{N_t,\ t = 1, \dots, T\}$ and $\{\tilde{N}_t,\ t = 1, \dots, T\}$ are sequences of i.i.d. random variables with PMF (probability mass function) P_N and $P_{\tilde{N}}$, respectively. These sequences are independent of each other and are also independent of the source output $\{X_t,\ t = 1, \dots, T\}$.

At each stage t, the encoder observes the output X_t of the source and the output $\tilde{Y}_{t-1}$ of the backward channel. It generates an encoded symbol Z_t using all its past observations, i.e., for $t = 1$,

$$Z_1 = c_1(X_1), \tag{2a}$$

and for $t = 2, \dots, T$,

$$Z_t = c_t(X^t, Z^{t-1}, \tilde{Y}^{t-1}), \tag{2b}$$

This encoded symbol is transmitted over the forward channel (1a) producing a channel output Y_t.

The receiver consists of a decoder and a memory. The content of the memory is denoted by M_t and takes values in a finite alphabet $\mathcal{M}$. At each stage t, the receiver generates an estimate $\hat{X}_t$ of the source taking values in $\hat{\mathcal{X}}$ as follows:

$$\hat{X}_1 = g_1(Y_1), \tag{3a}$$

and for $t = 2, \dots, T$,

$$\hat{X}_t = g_t(Y_t, M_{t-1}); \tag{3b}$$

it also generates a feedback symbol $\tilde{Z}_t$ as follows:

$$\tilde{Z}_1 = \tilde{c}_1(Y_1), \tag{4a}$$

and for $t = 2, \dots, T$,

$$\tilde{Z}_t = \tilde{c}_t(Y_t, M_{t-1}); \tag{4b}$$

furthermore, it updates the content of its memory as follows:

$$M_1 = l_1(Y_1), \tag{5a}$$

and for $t = 2, \dots, T$,

$$M_t = l_t(Y_t, M_{t-1}). \tag{5b}$$

The performance of the system is quantified by a uniformly bounded distortion function $\rho : \mathcal{X} \times \hat{\mathcal{X}} \to [0, \rho_{\max}]$, where $\rho_{\max} < \infty$. The distortion at time t is given by $\rho(X_t, \hat{X}_t)$.

The collection $C := (c_1, \dots, c_T)$ of encoding rules for the entire horizon is called an *encoding strategy*. Similarly, the collection $G := (g_1, \dots, g_T)$ of decoding rules is called a *decoding strategy*, the collection $\tilde{C} := (\tilde{c}_1, \dots, \tilde{c}_T)$ of feedback rules is called a *feedback strategy*, and the collection $L := (l_1, \dots, l_T)$ of memory update rules is called a *memory update strategy*. Further, the choice $(C, G, \tilde{C}, L)$ of communication rules for the entire horizon is called a *communication strategy*. The performance of a communication strategy is quantified by the expected total distortion under that strategy and is given by

$$\mathcal{J}_T(C, G, \tilde{C}, L) := \mathbb{E}\left\{ \sum_{t=1}^{T} \rho(X_t, \hat{X}_t) \,\middle|\, C, G, \tilde{C}, L \right\}. \tag{6}$$

We are interested in the following optimization problem:

Problem 1: Assume that the encoder and the receiver know the source statistics P_{X_1} and $P_{X_{t+1}|X_t}$, $t = 1, \dots, T$, the forward and backward channel functions h, $\tilde{h}$, the forward and the backward channel noise statistics P_N and $P_{\tilde{N}}$, the distortion functions ρ and the time horizon T. Choose a communication strategy $(C^*, G^*, \tilde{C}^*, L^*)$ that is optimal with respect to performance criterion of (6), i.e.,

$$\mathcal{J}_T(C^*, G^*, \tilde{C}^*, L^*) = \mathcal{J}_T^* := \min_{\substack{C \in \mathscr{C}^T \\ G \in \mathscr{G}^T \\ \tilde{C} \in \tilde{\mathscr{C}}^T \\ L \in \mathscr{L}^T}} \mathcal{J}_T(C, G, \tilde{C}, L), \tag{7}$$

where $\mathscr{C}^T := \mathscr{C}_1 \times \cdots \times \mathscr{C}_T$, $\mathscr{C}_t$ is the family of functions from $\mathcal{X}^t \times \tilde{\mathcal{Y}}^{t-1} \times \mathcal{Z}^{t-1}$ to $\mathcal{Z}$, $\mathscr{G}^T := \mathscr{G} \times \cdots \times \mathscr{G}$ (T-times), $\mathscr{G}$ is the family of functions from $\mathcal{Y} \times \mathcal{M}$ to $\hat{\mathcal{X}}$, $\tilde{\mathscr{C}}^T := \tilde{\mathscr{C}} \times \cdots \times \tilde{\mathscr{C}}$ (T-times), $\tilde{\mathscr{C}}$ is the family of functions from $\mathcal{Y} \times \mathcal{M}$ to $\tilde{\mathcal{Z}}$, $\mathscr{L}^T := \mathscr{L} \times \cdots \times \mathscr{L}$ (T-times), and $\mathscr{L}$ is the family of functions from $\mathcal{Y} \times \mathcal{M}$ to $\mathcal{M}$.

In Problem 1, we want to identify the optimal communication strategy for encoding the outputs of a first-order Markov source over a forward DMC, when the receiver can transmit back feedback messages to the encoder over a backward DMC. The receiver must decode in real-time (i.e., with zero-delay). Due to this real-time constraint on communication, asymptotic results from information theory, like source entropy, channel capacity, and rate, that are fundamental in asymptotic communication theory are not appropriate for real-time communication. As a continuation of our previous work [23]–[25], [30], [31], in this paper we present a framework different from Shannon's information theoretic setup to study real-time communication. We consider the real-time communication problem as an optimization problem. A globally optimal communication strategy always exists because there are finitely many communication strategies and we can always choose the one with the best performance. The number of possible time varying communication strategies are exponential in the size of the time horizon and the cardinality of the alphabets which makes a brute force search for an optimal strategy computationally intractable. So, a systematic approach to search for an optimal communication strategy is required. In this paper, we present one such systematic approach, called *sequential decomposition*, that can be used to sequentially determine optimal communication strategy by proceeding backwards in time. The resultant "simplified" optimization problem has linear complexity in the size of the time horizon but exponential complexity in the cardinality of the alphabets.

We proceed with the analysis of Problem 1 as follows. We first present the salient features of the problem and develop the concepts and notation needed for the rest of the paper. We then identify the qualitative/structural properties of optimal encoders and optimal decoders that hold for any choice of other components of the communication strategy. Next we use these qualitative properties to develop a methodology for systematically searching for optimal communication strategies.

B. Primitive Random Variables

In the rest of this paper we will be working with conditional probabilities, probability measures of probability measures, and σ-fields. To be precise in our analysis we need to define the probability space clearly. For that matter, we first define the primitive random variables of the system.

Let χ_t a $|\mathcal{X}|$-dimensional random vector defined as follows: for $x \in \mathcal{X}$,

$$\chi_t(x) := \mathbb{I}\,[X_t = x] \quad \text{and} \quad \chi_t := [\chi_t(1), \dots, \chi_t(|\mathcal{X}|)] \tag{8}$$

There is a one-to-one relation between X_t and χ_t and we can use χ_t to have a martingale representation (stochastic difference equation) for the Markov chain $\{X_t, t = 1, \dots, T\}$ (see [32]) given by

$$\chi_{t+1} = P^T_{X_{t+1}|X_t} \chi_t + \theta_t, \tag{9}$$

where $\{\theta_t,\ t = 1, \dots, T\}$ is a sequence of independent zero-mean random vectors. Since we have assumed that the noise in the forward and the backward channels is independent of the source output, the random variables $(\chi_1, \theta_1, \dots, \theta_T, N_1, \dots, N_T, \tilde{N}_1, \dots, \tilde{N}_T)$ are independent. These random variables are called the *primitive random variables*. We assume that all primitive random variables are defined on a common probability space $(\Omega, \mathfrak{F}, P)$. If the communication strategy is fixed, all system variables can be defined in terms of the primitive random variables, and are $(\Omega, \mathfrak{F}, P)$ measurable. In the sequel, all (random) variables are assumed to be defined on $(\Omega, \mathfrak{F}, P)$.

C. Problem Classification

Problem 1 is a sequential stochastic optimization problem as defined in [33]. To understand the sequential nature of the problem, we need to refine the notion of time. We call each step of the system a *stage*. For each stage, we consider four

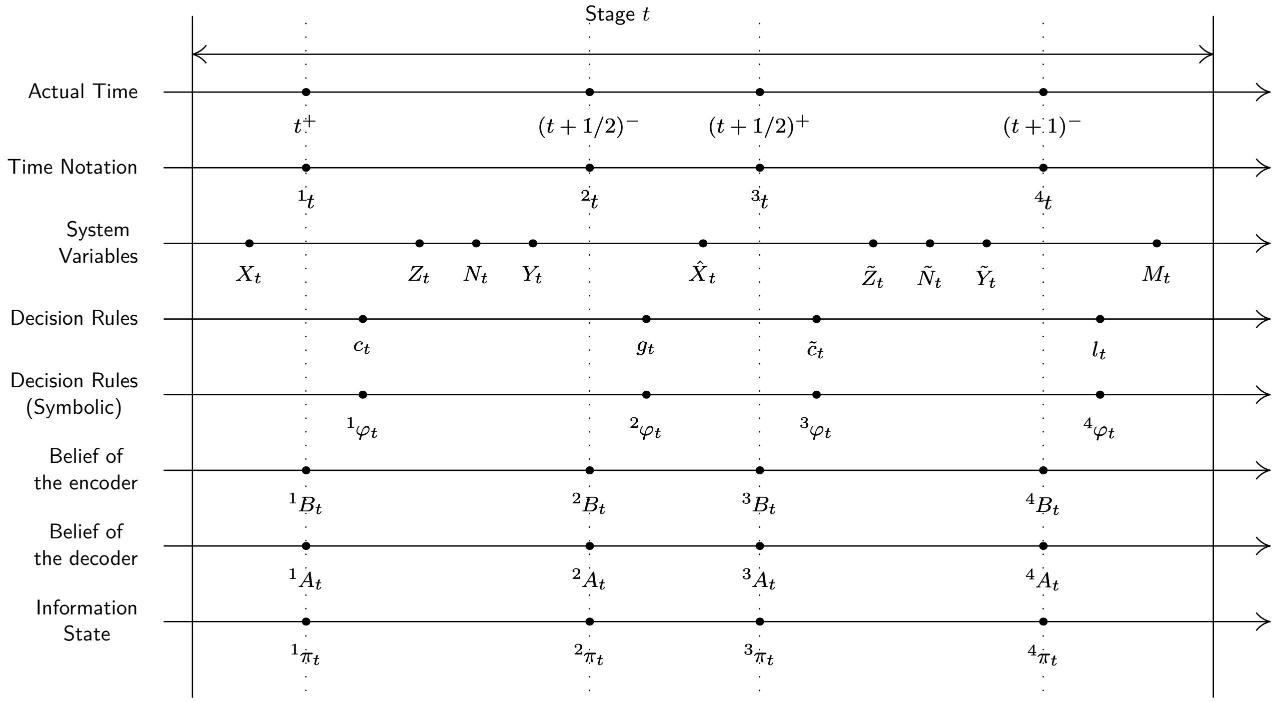

Fig. 2. Sequential ordering of different variables in the system

time instances:[1] t^+, $(t+1/2)^-$, $(t+1/2)^+$ and $(t+1)^-$. For the ease of notation, we will denote these time instances by ${}^1t$, ${}^2t$, ${}^3t$, and ${}^4t$, respectively. Assume that the system has four "agents", the encoder (agent 1), the decoder (agent 2), the feedback encoder (agent 3), and the memory update (agent 4), which act sequentially at ${}^1t$, ${}^2t$, ${}^3t$, and ${}^4t$, respectively. The order in which the random variables are generated in the system is illustrated in Figure II-B. Since the ordering of the decision makers can be done independently of the realization of the system variables, Property C of [34] is trivially satisfied and hence Problem 1 is a *causal sequential stochastic optimization* problem as defined in [33].

Problem 1 is a multi-agent problem where all agents have the same objective given by (7). Such problems are called team problems [35], and are further classified as static teams or dynamic teams on the basis of their information structure. In static teams, an agent's information is a function of primitive random variables only, while in dynamic teams, in general, an agent's information depends on the functional form of the decision rules of other agents. In Problem 1 the receiver's information depends on the functional form of the encoding rule. Thus Problem 1 is a dynamic team. Dynamic teams are, in general, functional optimization problems having a complex interdependence among the decision rules [36]. This interdependence leads to non-convex (in policy space) optimization problems that are hard to solve.

For the ease of notation, at time instances ${}^1t$, ${}^2t$, ${}^3t$, and ${}^4t$, we will denote the current decision rule by ${}^1\varphi_t$, ${}^2\varphi_t$, ${}^3\varphi_t$, and ${}^4\varphi_t$ and the past decision rules by ${}^1\varphi^{t-1}$, ${}^2\varphi^{t-1}$, ${}^3\varphi^{t-1}$, and ${}^4\varphi^{t-1}$, i.e.,

$$
\begin{aligned}
{}^1\varphi^{t-1} &:= (c^{t-1}, g^{t-1}, \tilde{c}^{t-1}, l^{t-1}), & {}^1\varphi_t &:= c_t, && \text{(10a)}\\
{}^2\varphi^{t-1} &:= (c^{t}, g^{t-1}, \tilde{c}^{t-1}, l^{t-1}), & {}^2\varphi_t &:= g_t, && \text{(10b)}\\
{}^3\varphi^{t-1} &:= (c^{t}, g^{t}, \tilde{c}^{t-1}, l^{t-1}), & {}^3\varphi_t &:= \tilde{c}_t, && \text{(10c)}\\
{}^4\varphi^{t-1} &:= (c^{t}, g^{t}, \tilde{c}^{t}, l^{t-1}), & {}^4\varphi_t &:= l_t. && \text{(10d)}
\end{aligned}
$$

D. The Notion of Information

The traditional information theoretic notions entropy and mutual information are asymptotic concepts which are not directly applicable to real-time communication problems. So, we first describe a decision theoretic notion of information. Recall that $(\Omega, \mathfrak{F}, P)$ is the probability space with respect to which all primitive random variables are defined. Suppose ${}^iO_t$ is the observation of agent i, at time ${}^it$, and ${}^i\varphi^{t-1}$ is the past decision rules of all agents. Since the problem is sequential, for any choice of ${}^i\varphi^{t-1}$, ${}^iO_t$ is measurable with respect to $\mathfrak{F}$. Furthermore, for any choice of ${}^i\varphi^{t-1}$, let $\sigma({}^iO_t; {}^i\varphi^{t-1})$ denote the smallest subfield of $\mathfrak{F}$ with respect to which ${}^iO_t$ is measurable. Then, the *information field* of agent i at time ${}^it$ is $\sigma({}^iO; {}^i\varphi^{t-1})$. Using this notion of information, we define variables that represent the information field at the encoder's and receiver's sites just before each agent acts on the system.

Definition 1: Let ${}^1E_t$, ${}^2E_t$, ${}^3E_t$, and ${}^4E_t$ denote the observation and ${}^1\mathfrak{E}_t$, ${}^2\mathfrak{E}_t$, ${}^3\mathfrak{E}_t$, and ${}^4\mathfrak{E}_t$ denote the *information field* at the encoder's site at time ${}^1t$, ${}^2t$, ${}^3t$, and ${}^4t$, respectively, i.e.,

$$
\begin{aligned}
{}^1E_t &:= (X^t, Z^{t-1}, \tilde{Y}^{t-1}), & {}^1\mathfrak{E}_t &:= \sigma({}^1E_t; {}^1\varphi^{t-1}), && \text{(11a)}\\
{}^2E_t &:= (X^t, Z^{t}, \tilde{Y}^{t-1}), & {}^2\mathfrak{E}_t &:= \sigma({}^2E_t; {}^2\varphi^{t-1}), && \text{(11b)}\\
{}^3E_t &:= (X^t, Z^{t}, \tilde{Y}^{t-1}), & {}^3\mathfrak{E}_t &:= \sigma({}^3E_t; {}^3\varphi^{t-1}), && \text{(11c)}\\
{}^4E_t &:= (X^t, Z^{t}, \tilde{Y}^{t}), & {}^4\mathfrak{E}_t &:= \sigma({}^4E_t; {}^4\varphi^{t-1}). && \text{(11d)}
\end{aligned}
$$

[1]The actual values of these time instances is not important; we just need four values in increasing order.

Further, let ${}^1R_t$, ${}^2R_t$, ${}^3R_t$, and ${}^4R_t$ denote the observation and ${}^1\mathfrak{R}_t$, ${}^2\mathfrak{R}_t$, ${}^3\mathfrak{R}_t$, and ${}^4\mathfrak{R}_t$ denote the *information field* at the receiver's site at time ${}^1t$, ${}^2t$, ${}^3t$, and ${}^4t$, respectively, i.e.,

$$ {}^1R_t := (M_{t-1}), \qquad {}^1\mathfrak{R}_t := \sigma({}^1R_t; {}^1\varphi^{t-1}), \tag{12a} $$
$$ {}^2R_t := (Y_t, M_{t-1}), \qquad {}^2\mathfrak{R}_t := \sigma({}^2R_t; {}^2\varphi^{t-1}), \tag{12b} $$
$$ {}^3R_t := (Y_t, M_{t-1}), \qquad {}^3\mathfrak{R}_t := \sigma({}^3R_t; {}^3\varphi^{t-1}), \tag{12c} $$
$$ {}^4R_t := (Y_t, M_{t-1}), \qquad {}^4\mathfrak{R}_t := \sigma({}^4R_t; {}^4\varphi^{t-1}). \tag{12d} $$

Let ${}^i\mathcal{E}_t$ and ${}^i\mathcal{R}$, $i = 1, \dots, 4$, denote the space of realizations of ${}^iE_t$ and ${}^iR$, respectively.

Problem 1 is a decentralized problem because, at any time t, the information fields at the encoder's site and the receiver's site are non-comparable, that is, ${}^1\mathfrak{E}_t \not\subseteq {}^1\mathfrak{R}_t$ and ${}^1\mathfrak{E}_t \not\supseteq {}^1\mathfrak{R}_t$; and similar relations hold between ${}^2\mathfrak{E}_t$ and ${}^2\mathfrak{R}_t$, between ${}^3\mathfrak{E}_t$ and ${}^3\mathfrak{R}_t$, and between ${}^4\mathfrak{E}_t$ and ${}^4\mathfrak{R}_t$. Thus, at no time during the evolution of the system does the encoder "know" exactly what is "known" to the receiver and vice-versa. *Hence the information in the system is decentralized*. Notice that the information fields at the encoder and the receiver are coupled through decision rules. ${}^1\mathfrak{E}_1$ and ${}^1\mathfrak{R}_1$ are known before the system starts operating. The choice of ${}^1\varphi_1$ determines ${}^2\mathfrak{E}_1$ and ${}^2\mathfrak{R}_1$, the choice of ${}^2\varphi_1$ determines ${}^3\mathfrak{E}_1$ and ${}^3\mathfrak{R}_1$, and so on. Thus, ${}^1\mathfrak{E}_t$ and ${}^1\mathfrak{R}_t$ are determined completely by ${}^1\mathfrak{E}_1$, ${}^1\varphi^{t-1}$ and ${}^1\mathfrak{R}_1$, ${}^1\varphi^{t-1}$, respectively. Thus, the information ${}^1\mathfrak{E}_t$ and ${}^1\mathfrak{R}_t$ is coupled through the past decision rules ${}^1\varphi^{t-1}$. Hence, Problem 1 has a *non-classical information structure* (see [37], [38]).

Note that the information at the encoder is nested while the information at the receiver is not. Formally, at the encoder we have ${}^1\mathfrak{E}_t \subseteq {}^2\mathfrak{E}_t \subseteq {}^3\mathfrak{E}_t \subseteq {}^4\mathfrak{E}_t \subseteq {}^1\mathfrak{E}_{t+1} \cdots$ and so on. So at any time the encoder remembers everything that it knew in the past. On the other hand, at the receiver we have ${}^1\mathfrak{R}_t \subseteq {}^2\mathfrak{R}_t \subseteq {}^3\mathfrak{R}_t \subseteq {}^4\mathfrak{R}_t$, but ${}^3\mathfrak{R}_t \not\subseteq {}^1\mathfrak{R}_{t+1}$. Thus, while updating its memory at ${}^3t$, the receiver forgets (or sheds) some information that it knew earlier. This shedding of information has an interesting consequence when we consider the evolution of receiver's beliefs in the next subsection.

A Technical Remark: There is a subtle difference between our definition and Witsenhausen's [34] definition of information field. In [34], information fields are defined on the product space $(\prod_\alpha \mathcal{U}_\alpha, \prod_\alpha \mathfrak{F}_\alpha)$, where α ranges over the set of all agents and $(\mathcal{U}_\alpha, \mathfrak{F}_\alpha)$ is the decision space of agent α (see [34] for details). In this paper, information field is defined on $(\Omega, \mathfrak{F})$, the probability space of all randomness in the system. Our definition allows us to define conditional expectations and conditional probabilities with respect to this information field (cf. [34, pp. 152]). What we call information field is the same as the induced field $\mathscr{I}_\alpha^\gamma$ in [34]. Unfortunately this field $\mathscr{I}_\alpha^\gamma$ was not given any name in [34]. We hope that with this clarification, our slight deviation in terminology from the literature will not cause any confusion.

E. Agent's Beliefs and their Evolution

Due to decentralization of information, it is important to characterize what one agent thinks about the other agent's observation, i.e., what the encoder "thinks" that the receiver "sees" and what the receiver "thinks" that the encoder "sees". This is captured by the encoder's belief about the observations of the receiver, and the receiver's belief about the observations of the encoder at time instances ${}^1t$, ${}^2t$, ${}^3t$, and ${}^4t$. These beliefs are given below.

Definition 2: Let ${}^1B_t$, ${}^2B_t$, ${}^3B_t$, and ${}^4B_t$ denote the encoder's belief about the receiver's observation at ${}^1t$, ${}^2t$, ${}^3t$, and ${}^4t$, respectively, i.e.,

$$ {}^1B_t({}^1r) := \Pr\left({}^1R_t = {}^1r \,\middle|\, {}^1\mathfrak{E}_t\right), \tag{13a} $$
$$ {}^2B_t({}^2r) := \Pr\left({}^2R_t = {}^2r \,\middle|\, {}^2\mathfrak{E}_t\right), \tag{13b} $$
$$ {}^3B_t({}^3r) := \Pr\left({}^3R_t = {}^3r \,\middle|\, {}^3\mathfrak{E}_t\right), \tag{13c} $$
$$ {}^4B_t({}^4r) := \Pr\left({}^4R_t = {}^4r \,\middle|\, {}^4\mathfrak{E}_t\right). \tag{13d} $$

Let ${}^i\mathcal{B} := \mathbb{P}\left\{{}^i\mathcal{R}\right\}$, $i = 1, \dots, 4$, denote the space of realizations of ${}^iB$.

Definition 3: Let ${}^1A_t$, ${}^2A_t$, ${}^3A_t$, and ${}^4A_t$ denote the receiver's belief about the encoder's observation at ${}^1t$, ${}^2t$, ${}^3t$, and ${}^4t$, respectively, i.e.,

$$ {}^1A_t({}^1e) := \Pr\left({}^1E_t = {}^1e \,\middle|\, {}^1\mathfrak{R}_t\right), \tag{14a} $$
$$ {}^2A_t({}^2e) := \Pr\left({}^2E_t = {}^2e \,\middle|\, {}^2\mathfrak{R}_t\right), \tag{14b} $$
$$ {}^3A_t({}^3e) := \Pr\left({}^3E_t = {}^3e \,\middle|\, {}^3\mathfrak{R}_t\right), \tag{14c} $$
$$ {}^4A_t({}^4e) := \Pr\left({}^4E_t = {}^4e \,\middle|\, {}^4\mathfrak{R}_t\right). \tag{14d} $$

Further, let $\hat{A}_t$ denote the receiver's belief about the source output at time instance ${}^2t$, i.e.,

$$ \hat{A}_t(x_t) := \Pr\left(X_t = x_t \,\middle|\, {}^2\mathfrak{R}_t\right). \tag{14e} $$

Let ${}^i\mathcal{A}_t := \mathbb{P}\left\{{}^i\mathcal{E}_t\right\}$, $i = 1, \dots, 4$, denote the space of realizations of ${}^iA_t$.

The sequential ordering of these beliefs is shown in Figure II-B. For any particular realization ${}^1e_t$ of ${}^1E_t$, and any arbitrary (but fixed) choice of ${}^1\varphi^{t-1}$, the realization ${}^1b_t$ of ${}^1B_t$ is a PMF on $\mathcal{M}$. If E_t is a random vector, then ${}^1B_t$ is a random vector belonging to $\mathbb{P}\{\mathcal{M}\}$, the space of PMFs on $\mathcal{M}$. Similar interpretations hold for ${}^2B_t$, ${}^3B_t$, ${}^4B_t$, ${}^1A_t$, ${}^2A_t$, ${}^3A_t$, and ${}^4A_t$.

The time evolution of these beliefs of the encoder and the receiver are coupled through their decision rules. Specifically,

Lemma 1: For each stage t, there exist deterministic functions ${}^1F$, ${}^2F$, and ${}^4F$ such that

$$ {}^1B_t = {}^1F({}^4B_{t-1}, l_{t-1}), \tag{15a} $$
$$ {}^2B_t = {}^2F({}^1B_t, Z_t), \tag{15b} $$
$$ {}^3B_t = {}^2B_t, \tag{15c} $$
$$ {}^4B_t = {}^4F({}^3B_t, \tilde{Y}_t, \tilde{c}_t). \tag{15d} $$

Lemma 2: For each stage t, there exists a deterministic function ${}^1K_t$ such that

$$ {}^1A_t = {}^1K_t({}^1A_1, M_{t-1}, c^{t-1}, \tilde{c}^{t-1}, l^{t-1}). \tag{16a} $$

Further, there exist functions, ${}^2K$, $\hat{K}$, and ${}^4K$ such that for each t,

$$ {}^2A_t = {}^2K({}^1A_t, Y_t, c_t), \tag{16b} $$
$$ \hat{A}_t = \hat{K}({}^2A_t), \tag{16c} $$
$$ {}^3A_t = {}^2A_t, \tag{16d} $$
$$ {}^4A_t = {}^4K({}^3A_t, \tilde{Z}_t). \tag{16e} $$

These lemmas are proved in Appendix A.

F. A Remark about Functional Dependencies

The belief evolution presented above has some interesting features which are a result of the functional coupling between the information fields of the encoder and the receiver. This can be illustrated by carefully analyzing the above belief evolutions. Consider the system at time ${}^1t$ and assume that the encoder and the receiver know the past decision rules ${}^1\varphi^{t-1}$. Using the results of Lemmas 1 and 2 the encoder can calculate ${}^1B_t$ and the decoder can calculate ${}^1A_t$. Now, at time ${}^1t$, the encoder generates the encoded symbol Z_t and the receiver receives the channel output Y_t. The encoder can update its belief using the encoded symbol (the decision) Z_t (see (15b)). The receiver, however, needs to know the encoding *rule* (the decision *rule*) c_t in order to update its belief (see (16b)). At time ${}^2t$, the receiver generates its estimate $\hat{X}_t$. Since decoding is an open loop problem, it does not influence the belief of either the encoder or the receiver (see (15c) and (16d)). Next at ${}^3t$, the receiver generates a feedback symbol $\tilde{Z}_t$ and the encoder receives $\tilde{Y}_t$. The receiver can update its belief using the feedback symbol (the decision) $\tilde{Z}_t$ (see (16d)). The encoder, however, needs to know the encoding *rule* (the decision *rule*) $\tilde{c}_t$ in order to update its belief (see (15c)). At ${}^4t$, the situation is a bit different at the receiver. The receiver needs to shed information while updating its memory, so ${}^3\mathfrak{R}_t \not\subseteq {}^1\mathfrak{R}_{t+1}$ and the receiver can not use just ${}^3A_t$ and M_t to generate ${}^1A_{t+1}$. The receiver needs to calculate ${}^1A_{t+1}$ *from scratch* using all past decision rules (see (16a)). The encoder still needs to know the memory update rule (the decision rule) l_t to update its belief.

Thus for decentralized problems with non-classical information structure, if at any time instant an agent, say i, makes the current decision and does not shed information it can update its belief using only the current decision; if the agent makes the current decision and sheds information, it needs to recalculate its belief from scratch using all the past decision rules. If agent i does not make the current decision, it needs to know the decision *rule* of the agent acting currently acting on the system in order to update its belief. Thus agents need to know the decision rules of other agents in order to update their beliefs. This gives rise to a functional coupling between the agents' beliefs. Contrast this from systems with classical information structure (centralized optimization problems) where the belief evolutions depends only on the current observation and/or current decision, and not on the current decision *rule*. It is this functional coupling among decision rules in decentralized problems with non-classical information structures that make the global optimization problem conceptually more difficult than the optimization problem in systems with classical information structure.

For the problem under consideration, before looking at the global optimization problem, we identify qualitative properties of optimal encoders and decoders.

III. Structural Properties

In this section, we provide qualitative properties of optimal encoders (respectively, decoders) that are true for every arbitrary but fixed decoding, feedback, and memory update strategies (respectively, encoding, feedback, and memory update strategies). These properties are subsequently used to develop a methodology for the determination of globally optimal communication strategies. They are also used to explain our methodology.

A. Structure of Optimal Real-Time Encoders

Theorem 1 (Structure of Optimal Encoders): Consider Problem 1 for any arbitrary (but fixed) decoding, feedback, and memory update strategies, $G = (g_1, \ldots, g_T)$, $\tilde{C} = (\tilde{c}_1, \ldots, \tilde{c}_T)$, and $L = (l_1, \ldots, l_T)$, respectively. Then there is no loss in optimality in restricting attention to encoding rules of the form

$$Z_t = c_t(X_t, {}^1B_t), \quad t = 2, \ldots, T. \tag{17}$$

Proof: We look at the problem from the encoder's point of view. Note that $\{X_t,\ t = 1, \ldots, T\}$ is a Markov process independent of the noise in the forward and the backward channels. This fact together with results of Lemma 2 implies that for any $x_{t+1} \in \mathcal{X}$, ${}^1b_{t+1} \in {}^1\mathcal{B}$, any realization $(x^t, {}^1b^t, z^t)$ of $(X^t, {}^1B^t, Z^t)$, and any choice of ${}^1\varphi^t$, we have

$$\begin{aligned}
&\Pr\left(X_{t+1} = x_{t+1}, {}^1B_{t+1} = {}^1b_{t+1} \,\middle|\, x^t, {}^1b^t, z^t, {}^1\varphi^t\right) \\
&\quad= \sum_{\substack{y_t \in \mathcal{Y} \\ m_{t-1} \in \mathcal{M} \\ \tilde{z}_t \in \tilde{\mathcal{Z}} \\ \tilde{y}_t \in \tilde{\mathcal{Y}}}} \Pr\left(x_{t+1}, {}^1b_{t+1}, \tilde{y}_t, \tilde{z}_t, y_t, m_{t-1} \,\middle|\, x^t, {}^1b^t, z^t, {}^1\varphi^t\right) \\
&\quad= \sum_{\substack{y_t \in \mathcal{Y} \\ m_{t-1} \in \mathcal{M} \\ \tilde{z}_t \in \tilde{\mathcal{Z}} \\ \tilde{y}_t \in \tilde{\mathcal{Y}}}} \Pr\left({}^1b_{t+1} \,\middle|\, x^{t+1}, {}^1b^t, z^t, \tilde{y}_t, \tilde{z}_t, y_t, m_{t-1}; {}^1\varphi^t\right) \\
&\qquad\qquad \times \Pr\left(x_{t+1} \,\middle|\, x^t, {}^1b^t, z^t, \tilde{y}_t, \tilde{z}_t, y_t, m_{t-1}; {}^1\varphi^t\right) \\
&\qquad\qquad \times \Pr\left(\tilde{y}_t \,\middle|\, x^t, {}^1b^t, z^t, \tilde{z}_t, y_t, m_{t-1}; {}^1\varphi^t\right) \\
&\qquad\qquad \times \Pr\left(\tilde{z}_t \,\middle|\, x^t, {}^1b^t, z^t, y_t, m_{t-1}; {}^1\varphi^t\right) \\
&\qquad\qquad \times \Pr\left(y_t \,\middle|\, x^t, {}^1b^t, z^t, m_{t-1}; {}^1\varphi^t\right) \\
&\qquad\qquad \times \Pr\left(m_{t-1} \,\middle|\, x^t, {}^1b^t, z^t; {}^1\varphi^t\right) \\
&\quad\overset{(a)}{=} \sum_{\substack{y_t \in \mathcal{Y} \\ m_{t-1} \in \mathcal{M} \\ \tilde{z}_t \in \tilde{\mathcal{Z}} \\ \tilde{y}_t \in \tilde{\mathcal{Y}}}} \mathbb{I}\left[{}^1b_{t+1} = {}^1F\left({}^4F\left({}^2F({}^1b_t, z_t), \tilde{y}_t, \tilde{c}_t\right), l_t\right)\right] \\
&\qquad\qquad \times P_{X_{t+1}|X_t}(x_{t+1} \mid x_t) \\
&\qquad\qquad \times P_{\tilde{N}}\left(\tilde{N}_t \in \tilde{\mathcal{N}} : \tilde{y}_t = \tilde{h}(\tilde{z}_t, \tilde{N}_t)\right) \\
&\qquad\qquad \times \mathbb{I}\left[\tilde{z}_t = \tilde{c}_t(y_t, m_{t-1})\right] \\
&\qquad\qquad \times P_N\left(N \in \mathcal{N} : y_t = h(z_t, N_t)\right) \\
&\qquad\qquad \times {}^1b_t(m_{t-1}) \\
&\quad=: \Pr\left(X_{t+1} = x_{t+1}, {}^1B_{t+1} = {}^1b_{t+1} \,\middle|\, x_t, {}^1b_t, z_t, \tilde{c}_t, l_t\right),
\end{aligned} \tag{18}$$

where (a) follows from Lemma 1 and the sequential order in which the random variables are generated. Thus for fixed feedback and memory update strategies, $\{(X_t, {}^1B_t),\ t = 1, \ldots, T\}$ is a controlled Markov process with control action Z_t. Further, the expected conditional instantaneous distortion can be written as

$$\begin{aligned}
&\mathbb{E}\left\{\rho(X_t, \hat{X}_t) \,\middle|\, {}^3\mathfrak{E}_t\right\} \\
&\quad= \sum_{\substack{y_t \in \mathcal{Y} \\ m_{t-1} \in \mathcal{M}}} \rho\left(X_t, g_t(y_t, m_{t-1})\right) \Pr\left(y_t, m_{t-1} \,\middle|\, {}^3\mathfrak{E}_t\right) \\
&\quad= \sum_{\substack{y_t \in \mathcal{Y} \\ m_{t-1} \in \mathcal{M}}} \rho\left(X_t, g_t(y_t, m_{t-1})\right) {}^3B_t(y_t, m_{t-1})
\end{aligned}$$

$$\overset{(b)}{=} \sum_{\substack{y_t \in \mathcal{Y} \\ m_{t-1} \in \mathcal{M}}} \rho\big(X_t, g_t(y_t, m_{t-1})\big)\, {}^2F({}^1B_t, Z_t)(y_t, m_{t-1})$$
$$=: \tilde{\rho}(X_t, {}^1B_t, Z_t, g_t) \tag{19}$$

where (b) follows from Lemma 1. Thus, the total expected distortion can be written as

$$\mathbb{E}\left\{ \sum_{t=1}^{T} \rho(X_t, \hat{X}_t) \,\middle|\, C, G, \tilde{C}, L \right\}$$
$$= \mathbb{E}\left\{ \sum_{t=1}^{T} \mathbb{E}\left\{ \rho(X_t, \hat{X}_t) \,\middle|\, {}^3\mathfrak{E}_t \right\} \,\middle|\, C, G, \tilde{C}, L \right\}$$
$$= \mathbb{E}\left\{ \sum_{t=1}^{T} \tilde{\rho}(X_t, {}^1B_t, Z_t, g_t) \,\middle|\, C, G, \tilde{C}, L \right\}. \tag{20}$$

Hence from the encoder's point of view, we have a perfectly observed controlled Markov process $\{(X_t, {}^1B_t),\ t = 1, \dots, T\}$ with control action Z_t and an instantaneous distortion $\tilde{\rho}(X_t, {}^1B_t, Z_t, g_t)$ (recall that G is fixed). From Markov decision theory [2, Chapter 6] we know that there is no loss of optimality in restricting attention to encoding rules of the form (17). ■

B. Structure of Optimal Real-Time Decoders

Theorem 2 (Structure of Optimal Decoders): Consider Problem 1 for any arbitrary (but fixed) encoding, feedback, and memory update strategies, $C = (c_1, \dots, c_T)$, $\tilde{C} = (\tilde{c}_1, \dots, \tilde{c}_T)$, and $L = (l_1, \dots, l_T)$, respectively. Then there is no loss in optimality in restricting attention to decoding rules of the form

$$\hat{X}_t = \hat{g}(\hat{A}_t) := \arg\min_{\hat{x} \in \hat{\mathcal{X}}} \sum_{x \in \mathcal{X}} \rho(x, \hat{x}) \hat{A}_t(x). \tag{21}$$

Proof: We look at the problem from the decoder's point of view. Since decoding is a filtering problem, minimizing the total distortion $\mathcal{J}_T(C, G, \tilde{C}, L)$ is equivalent to minimizing the conditional expected instantaneous distortion $\mathbb{E}\left\{\rho(X_t, \hat{X}_t) \,\middle|\, {}^2\mathfrak{R}_t\right\}$ for each time t. This conditional expected instantaneous distortion can be written as

$$\mathbb{E}\left\{\rho(X_t, \hat{X}_t) \,\middle|\, {}^2\mathfrak{R}_t\right\} = \sum_{x_t \in \mathcal{X}} \rho(x_t, \hat{X}_t) \Pr\left(x_t \,\middle|\, {}^2\mathfrak{R}_t\right)$$
$$= \sum_{x_t \in \mathcal{X}} \rho(x_t, \hat{X}_t) \hat{A}_t(x_t) \tag{22}$$

and is minimized by the decoding rule given in (21). ■

C. Remarks on the Assumption of Finite Memory at the Receiver

In the model of this paper we have assumed that the receiver has finite memory. Due to this assumption, the space ${}^1\mathcal{R}$ of realizations of ${}^1R_t$ does not depend on t. Consequently, the space ${}^1\mathcal{B}$ of realization of ${}^1B_t$ does not depend on t. Thus, qualitative properties of the form (17) for the encoding rule imply that at each time we can search for optimal encoding rules belonging to a space that does not depend on t. Furthermore, for systems with large horizon T, encoding rules of the form (17) can be easier to implement than encoding rules of the form (2).

Even if the receiver had no restriction on the size of its memory, the result of Theorem 1 will be true. However, the space ${}^1\mathcal{B}_t$ of the realization of ${}^1B_t$ will increase in size with t. So encoding rules of the form (17) would not be easier to implement than those of the form (2).

In Section V we present a methodology for the sequential decomposition of the global optimization problem. The information states (sufficient statistics) for this sequential decomposition have a nice compact representation because encoding rules of the form (17) belong to a space that does not increase with time. If the decoder had no memory restriction, these information states would belong to spaces that increase with time.

Thus, the assumption of finite memory at the receiver is not necessary to derive the structural properties and sequential decomposition for the finite horizon problem considered in this paper; this assumption makes the structural results useful for implementing the encoding rules; it also makes it easier to present the sequential decomposition. However, if we want to extend the sequential decomposition presented in this paper to infinite horizon problems, the assumption of finite memory at the receiver becomes necessary. Even though we do not treat the infinite horizon problem in this paper, for the ease of presentation we have assumed finite memory at the receiver.

IV. Comparison with Previous Results in Real-Time Communication

As mentioned in the Introduction there are three different conceptual approaches to real-time (zero-delay) communication problems, namely (i) performance bounds of finite-delay or real-time communication systems; (ii) real-time encoding and decoding of individual sequences; and (iii) real-time encoding and decoding of Markov sources. The approach taken in this paper falls under the last category. For real-time encoding and decoding of Markov sources, three different channel models have been considered: noiseless channels in [16]; noisy DMC with noiseless feedback in [19] and noisy AWGN (additive white Gaussian noise) channels with noiseless feedback [20]–[22]; and noisy DMC with no feedback [23], [24]. These problems/models are special cases of Problem 1 for appropriate choices of feedback strategies, and forward and backward channels. In this section, we rederive structural results for these communication systems using the structural results of Section III. This shows that the results this paper subsume all existing structural results for real-time encoding-decoding of Markov sources and thus presents a unified framework to study these problems.

A. Real-Time Noiseless Communication

Consider the real-time source coding problem investigated in [16], which is shown in Figure 3. This problem can be considered as a special case of Problem 1 by making the following simplifications in the model considered in Section II-A.

Assume that the forward channel is noiseless, i.e.,

$$Y_t := h_t(Z_t, N_t) = Z_t, \quad \text{and} \quad \mathcal{Y} = \mathcal{Z}. \tag{23}$$

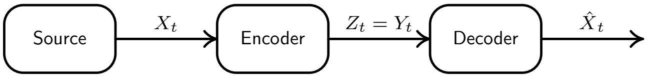

Fig. 3. A real-time source coding problem

Since the encoder knows output of the channel, the choice of feedback strategy $\tilde{C}$ and backward channel $\tilde{h}$ in the model of Section II-A does not matter. For completeness, assume that the backward channel is noiseless and the feedback strategy is to send back the output of the forward channel, i.e.,

$$\tilde{Z}_t \coloneqq \tilde{c}_t(Y_t, M_{t-1}) = Y_t, \quad \text{and} \quad \tilde{\mathcal{Z}} = \mathcal{Y}, \tag{24}$$

and

$$\tilde{Y}_t \coloneqq \tilde{h}_t(\tilde{Z}_t, \tilde{N}_t) = Z_t, \quad \text{and} \quad \tilde{\mathcal{Y}} = \tilde{\mathcal{Z}}. \tag{25}$$

Since the forward channel is noiseless, i.e. $Y_t = Z_t$, we can denote the encoded symbol by Y_t. Thus, using (23), (24) and (25) we can be write (2) as

$$Y_1 = c_1(X_1), \tag{26a}$$

and for $t = 2, \dots, T$,

$$Y_t = c_t(X^t, Y^{t-1}). \tag{26b}$$

The receiver operates according to (3) and (5) and the performance of a design is given by (6). The system model given by (26), (3), (5) and (6) is identical to the real-time source coding problem considered in [16]. For this model, ${}^1\mathfrak{R}_t \subseteq {}^1\mathfrak{E}_t$, ${}^3\mathfrak{E}_t \subseteq {}^3\mathfrak{R}_t$, ${}^2\mathfrak{R}_t \subseteq {}^2\mathfrak{E}_t$, and ${}^4\mathfrak{R}_t \subseteq {}^4\mathfrak{E}_t$. Thus, the model has a nested information structure. For fixed decoding and memory update strategies, the information structure at the encoder is a classical one, and the beliefs of the encoder (given by Definition 2) simplify as follows:

$${}^1B_t(m_{t-1}) = \mathbb{I}\left[m_{t-1} = {}^1l_{t-1}(Y^{t-1}, l^{t-1})\right], \tag{27a}$$

$$\begin{aligned} {}^2B_t(y_t, m_{t-1}) &= {}^3B_t(y_t, m_{t-1}) = {}^4B_t(y_t, m_{t-1}) \\ &= \mathbb{I}\left[y_t = Y_t\right] \mathbb{I}\left[m_{t-1} = {}^1l_{t-1}(Y^{t-1}, l^{t-1})\right], \end{aligned} \tag{27b}$$

where

$${}^1l_{t-1}(Y^{t-1}, l^{t-1}) = l_{t-1}\big(Y_{t-1}, l_{t-2}(Y_{t-2}, \dots, l_1(Y_1)\dots)\big). \tag{28}$$

Thus, belief ${}^1B_t$ is a random variable belonging to $\mathbb{P}\{M\}$ with unit mass on M_{t-1}. This means that the encoder knows the realization of M_{t-1} and implies that there is a one-to-one correspondence between ${}^1B_t$ and M_{t-1}. By a similar argument, there is a one-to-one correspondence between ${}^2B_t$ and (Y_t, M_{t-1}), between ${}^3B_t$ and (Y_t, M_{t-1}), and between ${}^4B_t$ and (Y_t, M_{t-1}). Due to these one-to-one correspondences the result of Theorem 1 can be simplified as follows:

Corollary 1: Consider Problem 1 for noiseless forward and backward channels, feedback strategy given by (24), and arbitrary (but fixed) decoding and memory update strategies, $G = (g_1, \dots, g_T)$ and $L = (l_1, \dots, l_T)$, respectively. Then there is no loss in optimality in restricting attention to encoding rules of the form

$$Z_t = c_t(X_t, M_{t-1}), \quad t = 2, \dots, T. \tag{29}$$

This result is identical to [16, Lemma 4, pp. 1446] which is the main result of [16] for first order Markov sources.

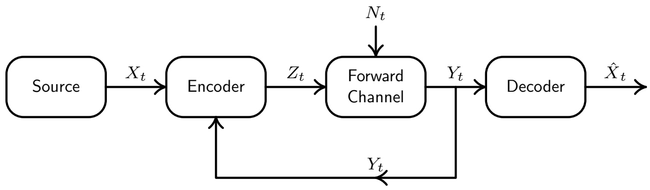

Fig. 4. A real-time communication system with noiseless feedback

B. Real-Time Noisy Communication with Noiseless Feedback

Consider the real-time communication problem with noiseless feedback investigated in [19], which is shown in Figure 4. This problem can be considered as a special case of Problem 1 by making the following simplifications in the model considered in Section II-A.

Assume that the backward channel is noiseless and that the feedback strategy is to send back the output of the forward, i.e.,

$$\tilde{Z}_t \coloneqq \tilde{c}_t(Y_t, M_{t-1}) = Y_t, \quad \text{and} \quad \tilde{\mathcal{Z}} = \mathcal{Y}, \tag{30}$$

and

$$\tilde{Y}_t \coloneqq \tilde{h}(\tilde{Z}_t, \tilde{N}_t) = \tilde{Z}_t, \quad \text{and} \quad \tilde{\mathcal{Y}} = \tilde{\mathcal{Z}}. \tag{31}$$

Relations (30) and (31) imply that $\tilde{Y}_{t-1} = Y_{t-1}$. Thus, the encoder can be written as

$$Z_1 = c_1(X_1), \tag{32a}$$

and for $t = 2, \dots, T$

$$Z_t = c_t(X^t, Z^{t-1}, Y^{t-1}). \tag{32b}$$

The receiver operates according to (3) and (5) and the performance of a design is given by (6). The system model given by (32), (3), (5) and (6) is identical to the real-time coding problem with noiseless feedback considered in [19]. For this model, ${}^1\mathfrak{R}_t \subseteq {}^1\mathfrak{E}_t$, ${}^2\mathfrak{R}_t \subseteq {}^2\mathfrak{E}_t$, ${}^3\mathfrak{R}_t \subseteq {}^3\mathfrak{E}_t$, and ${}^4\mathfrak{R}_t \subseteq {}^4\mathfrak{E}_t$. Thus, the model has a nested information structure. For fixed decoding and memory update strategies, the information structure at the encoder is a classical one, and the beliefs of the encoder (given by Definition 2) simplify according to (27). Further, by the same argument as in Section IV-A, there is a one-to-one correspondence between ${}^1B_t$ and M_{t-1}, between ${}^2B_t$ and (Y_t, M_{t-1}), between ${}^3B_t$ and (Y_t, M_{t-1}), and between ${}^4B_t$ and (Y_t, M_{t-1}). Due to these one-to-one correspondences the result of Theorem 1 can be simplified as follows:

Corollary 2: Consider Problem 1 for noiseless backward channel, feedback strategy given by (30), and arbitrary (but fixed) decoding and memory update strategies, $G = (g_1, \dots, g_T)$ and $L = (l_1, \dots, l_T)$, respectively. Then there is no loss in optimality in restricting attention to encoding rules of the form

$$Z_t = c_t(X_t, M_{t-1}), \quad t = 2, \dots, T. \tag{33}$$

The receiver model considered in [19] is more general than the model we consider in this paper; in [19], the receiver can either have a finite memory or no there is no restriction on the size of the memory. Corollary 2 is identical to [19, Theorem 1] and the structure of optimal decoder (Theorem 2) is similar

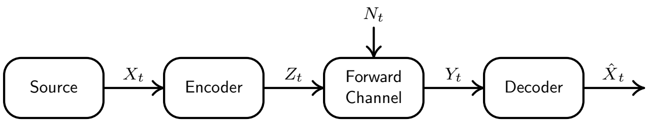

Fig. 5. A real-time communication system with no feedback

to [19, Lemma 1] when the model of [19] is restricted to have a finite memory.

Another variation of this problem that has been investigated in the literature is when the source output is scalar Gauss Markov process, the channel noise is AWGN (additive white Gaussian noise), the distortion measure is quadratic, and the decoder has no memory limitation. This problem was considered for discrete time in [21] and for continuous time in [20], [22] and it was shown that linear encoding and decoding strategies are optimal. These results rely on the special properties of LQG (linear quadratic Gaussian) systems can be thought of as further refinement of the structural property given by Corollary 2.

C. Real-Time Noisy Communication with no Feedback

Consider the real-time communication problem with no feedback investigated in [23], which is shown in Figure 4. This problem can be considered as a special case of Problem 1 by making the following simplifications in the model considered in Section II-A.

Assume the backward channel does not provide any valuable information to the encoder, that is

$$\tilde{Y}_t := \tilde{h}_t(\tilde{Z}_t, \tilde{N}_t) = \tilde{N}_t, \quad \text{and} \quad \tilde{\mathcal{Y}} = \tilde{\mathcal{N}}. \tag{34}$$

Since, the output of the backward channel is independent of all random variables in the system, it can be ignored. Thus, the encoder is of the form:

$$Z_1 = c_1(X_1), \tag{35a}$$

and for $t = 2, \dots, T$

$$Z_t = c_t(X^t, Z^{t-1}). \tag{35b}$$

Thus, the information available at the encoder (given by Definition 1) changes to:

$$^1E_t := (X^t, Z^{t-1}), \tag{36a}$$
$$^2E_t := (X^t, Z^t), \tag{36b}$$
$$^3E_t := (X^t, Z^t), \tag{36c}$$
$$^4E_t := (X^t, Z^t). \tag{36d}$$

Since the feedback symbol $\tilde{Z}_t$ is ignored, the choice of feedback strategy is immaterial. The receiver operates according to (3) and (5) and the performance of a design is given by (6). The system model given by (35), (3), (5) and (6) is identical to the real-time coding problem with no feedback considered in [23], [24], [39]. For this model, ${}^1\mathfrak{R}_t \not\subseteq {}^1\mathfrak{E}_t$, ${}^1\mathfrak{E}_t \not\subseteq {}^1\mathfrak{R}_t$, and similar relations hold between ${}^2\mathfrak{R}_t$ and ${}^2\mathfrak{E}_t$, between ${}^3\mathfrak{R}_t$ and ${}^3\mathfrak{E}_t$, and between ${}^4\mathfrak{R}_t$ and ${}^4\mathfrak{E}_t$. Thus, the model has a non-classical information structure. The belief of the encoder (given by Definition 2) are now defined by conditioning on the information at the encoder given by (36). With this modification, the belief ${}^1B_t$ is identical to $P_{W_{t-1}}$ defined in [23, Eq. (13)], the structure of optimal encoder (Theorem 1) is identical to [23, Theorem 1] and the structure of optimal decoder (Theorem 2) is identical to [23, Theorem 2].

D. Comparison of Different Structural Results

We have shown that structural results of Section III subsume all previously known structural results for real-time encoding-decoding of Markov sources. There are two categories of communication systems: (i) Systems with nested information structures (models of Sections IV-A and IV-B), and (ii) systems with non-nested information structures (models of Sections IV-C and II-A). The qualitative properties of optimal encoders within each category are similar.

Two proof methodologies have been employed in the literature to derive qualitative properties of optimal encoders: an *interchange argument* used in [16] and [23] and; *controlled Markov process* argument used in [19] and the alternative proof in [23]. In this paper we have used controlled Markov process argument to derive qualitative properties of optimal encoders (Theorem 1). An interchange argument, along the lines of one presented in [23], can also be used to prove Theorem 1, but we do not present it here.

V. Determining Globally Optimal Communication Strategies

A. Information States

The key step in obtaining a sequential decomposition is to identify an *information state sufficient for performance evaluation* (also called a *sufficient statistic for control*). See [24] for a discussion on the properties of such information states. In order to identify appropriate information states for Problem 1, we first use results of Theorem 1 to reformulate Problem 1 in a slightly different manner.

Let $\hat{\mathscr{C}}$ denote the space of functions from $\mathcal{X} \times \mathbb{P}\{\mathcal{M}\}$ to $\mathcal{Z}$. The result of Theorem 1 states that instead of choosing an encoding rule from the space $\mathscr{C}_t$ at time t, we can choose an encoding rule from the space $\hat{\mathscr{C}}$. Therefore, we have

Corollary 3: The optimal performance $\mathcal{J}_T^*$ given by (7) can be determined by

$$\mathcal{J}_T^* := \min_{\substack{C \in \hat{\mathscr{C}}^T \\ G \in \mathscr{G}^T \\ \tilde{C} \in \tilde{\mathscr{C}}^T \\ L \in \mathscr{L}^T}} \mathcal{J}_T(C, G, \tilde{C}, L), \tag{37}$$

where $\hat{\mathscr{C}}^T := \hat{\mathscr{C}} \times \cdots \times \hat{\mathscr{C}}$ (T-times), and $\mathscr{G}^T$ and $\mathscr{L}^T$ are defined as before.

Hence, in Problem 1 rather than choosing a communication strategy $(C^*, G^*, \tilde{C}^*, L^*)$ belonging to $(\mathscr{C}^T \times \mathscr{G}^T \times \tilde{\mathscr{C}}^T \times \mathscr{L}^T)$ to minimize (7) we can choose a communication strategy $(C^*, G^*, \tilde{C}^*, L^*)$ belonging to $(\hat{\mathscr{C}}^T \times \mathscr{G}^T \times \tilde{\mathscr{C}}^T \times \mathscr{L}^T)$ to minimize (37). Notice that the domain of an encoding rule belonging to $\mathscr{C}_t$ increases with t, while the domain of an encoding rule belonging to $\hat{\mathscr{C}}$ does not depend on t. Hence, using the structural results of Theorem 1 we can reformulate Problem 1 such that the encoding rules at each time have to be chosen from a time-invariant space. For this reformulation, an information state sufficient for performance evaluation is given below.

Definition 4: Define ${}^1\pi_t$, ${}^2\pi_t$, ${}^3\pi_t$, and ${}^4\pi_t$ as follows:

$$ {}^1\pi_t := \Pr\left(X_t, {}^1R_t, {}^1B_t \,\middle|\, {}^1\phi^{t-1}\right), \quad (38a)$$
$$ {}^2\pi_t := \Pr\left(X_t, {}^2R_t, {}^2B_t \,\middle|\, {}^2\phi^{t-1}\right), \quad (38b)$$
$$ {}^3\pi_t := \Pr\left(X_t, {}^3R_t, {}^3B_t \,\middle|\, {}^3\phi^{t-1}\right), \quad (38c)$$
$$ {}^4\pi_t := \Pr\left(X_t, {}^4R_t, {}^4B_t \,\middle|\, {}^4\phi^{t-1}\right). \quad (38d)$$

Let ${}^i\Pi$, $i = 1, \ldots, 4$, denote the space of probability measures on $(\mathcal{X} \times {}^i\mathcal{R} \times {}^i\mathcal{B})$. Then ${}^i\pi_t$ takes values in ${}^i\Pi$.

The above definitions are to be interpreted as follows. Let $(\Omega, \mathfrak{F}, P)$ denote the probability space with respect to which all primitive random variables are defined. For any agent i, $i = 1, \ldots, 4$, and any choice ${}^i\phi^{t-1}$ of past decision rules, the variables ${}^iE_t$ and ${}^iR_t$, and the beliefs ${}^iB_t$ and ${}^iA_t$ are $\mathfrak{F}$-measurable. Thus, for any choice of ${}^i\phi^{t-1}$, $(X_t, {}^iR_t, {}^iB_t)$ is $\mathfrak{F}$-measurable. ${}^i\pi_t$ is the corresponding induced measure on $(\mathcal{X} \times {}^i\mathcal{R} \times {}^i\mathcal{B})$.

The above-defined ${}^i\pi_t$, $i = 1, \ldots, 4$ are related as follows:

Lemma 3: For encoding rules of the form (17), ${}^1\pi_t$, ${}^2\pi_t$, ${}^3\pi_t$, and ${}^4\pi_t$ are information states for the encoder, decoder, feedback encoder, and memory update, respectively, i.e.,

1) there exist transformations ${}^1Q$, ${}^3Q$, and ${}^4Q$ such that

$$ {}^2\pi_t = {}^1Q(c_t)\, {}^1\pi_t, \quad (39a)$$
$$ {}^3\pi_t = {}^2\pi_t, \quad (39b)$$
$$ {}^4\pi_t = {}^3Q(\tilde{c}_t)\, {}^3\pi_t, \quad (39c)$$
$$ {}^1\pi_{t+1} = {}^4Q(l_t)\, {}^4\pi_t. \quad (39d)$$

These transformation are linear in the corresponding ${}^i\pi$, $i = 1, 3, 4$.

2) the expected instantaneous cost can be expressed as

$$ \mathbb{E}\left\{\rho(X_t, \hat{X}_t) \,\middle|\, {}^2\phi^{t-1}, g_t\right\} = \hat{\rho}({}^2\pi_t, g_t) \quad (40)$$

where $\hat{\rho}(\cdot)$ is a deterministic function concave in ${}^2\pi$.

This is proved in Appendix B.

Using this result, the performance criterion of (6) can be rewritten as

$$ \begin{aligned} \mathcal{J}_T(C, G, \tilde{C}, L) &= \mathbb{E}\left\{\sum_{t=1}^{T} \rho(X_t, \hat{X}_t) \,\middle|\, C, G, \tilde{C}, L\right\} \\ &\overset{(a)}{=} \sum_{t=1}^{T} \mathbb{E}\left\{\rho(X_t, \hat{X}_t) \,\middle|\, {}^2\phi^{t-1}, g_t\right\} \\ &\overset{(b)}{=} \sum_{t=1}^{T} \hat{\rho}({}^2\pi_t, g_t) \end{aligned} \quad (41)$$

where (a) follows from the sequential ordering of system variables and (b) follows from Lemma 3.

B. An Equivalent Optimization Problem

Consider a centralized deterministic optimization problem with state space alternating between ${}^1\Pi$, ${}^2\Pi$, ${}^3\Pi$, and ${}^4\Pi$ and action space alternating between $\hat{\mathscr{C}}, \mathscr{G}, \tilde{\mathscr{C}}$ and $\mathscr{L}$. The system dynamics are given by (39) and at each stage t the decision rules c_t, g_t, $\tilde{c}_t$, and l_t are determined according to *meta-rules* ${}^1\Delta_t$, ${}^2\Delta_t$, ${}^3\Delta_t$, and ${}^4\Delta_t$, where ${}^1\Delta_t$ is a function from ${}^1\Pi$ to $\hat{\mathscr{C}}$, ${}^2\Delta_t$ is a function from ${}^2\Pi$ to $\mathscr{G}$, ${}^3\Delta_t$ is a function from ${}^3\Pi$ to $\tilde{\mathscr{C}}$, and ${}^4\Delta_t$ is a function from ${}^4\Pi$ to $\mathscr{L}$. Thus the system equations (39) can be written as

$$ c_t = {}^1\Delta_t({}^1\pi_t), \qquad {}^2\pi_t = {}^1Q(c_t)\, {}^1\pi_t, \quad (42a)$$
$$ g_t = {}^2\Delta_t({}^2\pi_t), \qquad {}^3\pi_t = {}^2\pi_t, \quad (42b)$$
$$ \tilde{c}_t = {}^3\Delta_t({}^3\pi_t), \qquad {}^4\pi_t = {}^3Q(\tilde{c}_t)\, {}^3\pi_t, \quad (42c)$$
$$ l_t = {}^4\Delta_t({}^4\pi_t), \qquad {}^1\pi_{t+1} = {}^4Q(l_t)\, {}^4\pi_t. \quad (42d)$$

The initial state ${}^1\pi_1 = P_{X_1}$ is given. At each stage an instantaneous cost $\hat{\rho}({}^2\pi_t, g_t)$ is incurred. The choice $({}^1\Delta_1, {}^2\Delta_1, {}^3\Delta_1, {}^4\Delta_1, \ldots, {}^1\Delta_T, {}^2\Delta_T, {}^3\Delta_T, {}^4\Delta_T)$ is called a *meta-design* and denoted by Δ^T. The performance of a meta-design is given by the total cost incurred by that meta-design, i.e.,

$$ \mathcal{J}_T(\Delta^T \mid {}^1\pi_1) = \sum_{t=1}^{T} \hat{\rho}({}^2\pi_t, g_t). \quad (43)$$

Now consider the following optimization problem:

Problem 2: Consider the dynamic system (42) with known transformations ${}^1Q$, ${}^3Q$, and ${}^4Q$. The initial state ${}^1\pi_1$ is given. Determine a meta-design Δ^T to minimize the total cost given by (43).

C. The Global Optimization Algorithm

Observe that for any initial state ${}^1\pi_1$, a choice of meta-design Δ^T determines a design $(C, G, \tilde{C}, L)$ through (42). Relation (41) implies that the expected distortion under design $(C, G, \tilde{C}, L)$, given by (6), is equal to the cost under meta-design Δ^T given by (43). Thus, if the transformation ${}^1Q$, ${}^3Q$, and ${}^4Q$ in Problem 2 are the same as those in Lemma 3, an optimal meta-design for Problem 2 determines an optimal design for Problem 1. Problem 2 is a classical centralized deterministic control problem and optimal meta-designs can be determined as follows:

Theorem 3 (Global Optimization Algorithm): An optimal meta-design $\Delta^{*,T}$ for Problem 2, and consequently an optimal design $(C^*, G^*, \tilde{C}^*, L^*)$ for Problem 1 can be determined as follows. For any ${}^1\pi \in {}^1\Pi$, ${}^2\pi \in {}^2\Pi$, ${}^3\pi \in {}^3\Pi$, and ${}^4\pi \in {}^4\Pi$, define the following functions:

$$ {}^1V_{T+1}({}^1\pi) = 0, \quad (44a)$$

and for $t = 1, \ldots, T$

$$ {}^1V_t({}^1\pi) = \inf_{c \in \hat{\mathscr{C}}} {}^2V_t\left({}^1Q(c)\, {}^1\pi\right), \quad (44b)$$
$$ {}^2V_t({}^2\pi) = \min_{g \in \mathscr{G}} \left[\hat{\rho}({}^2\pi, g) + {}^3V_t({}^2\pi)\right], \quad (44c)$$
$$ {}^3V_t({}^3\pi) = \min_{\tilde{c} \in \tilde{\mathscr{C}}} {}^4V_t\left({}^3Q(\tilde{c})\, {}^3\pi\right), \quad (44d)$$
$$ {}^4V_t({}^4\pi) = \min_{l \in \mathscr{L}} {}^1V_{t+1}\left({}^4Q(l)\, {}^4\pi\right). \quad (44e)$$

The arg min (or arg inf) at each step determines the optimal meta-design $\Delta^{*,T}$. After an optimal meta-design has been determined, an optimal design $(C^*, G^*, \tilde{C}^*, L^*)$ can be determined through (42). Furthermore, the optimal performance is given by

$$ \mathcal{J}_T^* = {}^1V_1({}^1\pi_1). \quad (45)$$

Proof: This is a standard result, see [2, Chapter 2]. ■

The functions ${}^iV_t$, $i = 1, \dots, 4$, are called *value functions*. They represent the minimum expected future distortion that the system in state ${}^i\pi$ will incur from time ${}^it$ onwards. We have the following result.

Theorem 4 (Concavity of Value Functions): The value functions ${}^iV_t$, $i = 1, \dots, 4$, $t = 1, \dots, T$, given by (44) are concave in the corresponding ${}^i\pi$.

Proof: Recall that ${}^1Q$, ${}^3Q$, and ${}^4Q$ are linear in ${}^i\pi$ and $\hat{\rho}(\cdot)$ is concave in ${}^2\pi$. The result of the theorem follows from the fact that concavity is maintained under composition of a concave function with a linear transformation, summation of concave functions, and point-wise minimum/infimum of a concave function. A detailed proof is presented in Appendix C. ■

Observe that the four step T-stage sequential decomposition of (44) can be combined into a one-step T-stage sequential decomposition

$$
{}^1V_t({}^1\pi) = \inf_{\substack{c \in \mathscr{C} \\ g \in \mathscr{G} \\ \tilde{c} \in \tilde{\mathscr{C}} \\ l \in \mathscr{L}}} \Big[\hat{\rho}({}^1Q(c)\,{}^1\pi, g) + {}^1V_{t+1}\big(({}^4Q(l) \circ {}^3Q(\tilde{c}) \circ {}^1Q(c))\,{}^1\pi\big) \Big]. \tag{46}
$$

which is a deterministic dynamic program in function space. We present the finer decomposition in Theorem 3, since the finer decomposition given by (44) has a smaller search space than the more compact decomposition given by (46).

D. Computational Issues

The nested optimality equations of (44) are computationally difficult to solve for two reasons. The first reason is that the information states ${}^i\pi$, $i = 1, \dots, 4$, belong to an uncountable space ${}^i\Pi$. Hence, to solve (44) we need to discretize ${}^i\Pi$. Most discretization techniques result in an exponential growth of the number of points as we approximate the value function more closely, leading to the so called "curse of dimensionality". However, there are randomized techniques for discretization that avoid such an exponential growth (see [40], [41]). The bigger obstacle in efficiently solving (44) is that each step of (44) involves solving a non-convex functional optimization problem. For example, consider the optimization problem at ${}^4t$. The function ${}^1V_{t+1}\big({}^4Q(l)\,{}^4\pi\big)$ is concave in ${}^4\pi$, but it is neither concave nor convex in l, and we need to minimize ${}^1V_{t+1}\big({}^4Q(l)\,{}^4\pi\big)$ over all choices of l. So, the only way to do this is to evaluate ${}^1V_{t+1}\big({}^4Q(l)\,{}^4\pi\big)$ over all choices of l. There are $|\mathcal{M}|^{|\mathcal{Y}||\mathcal{M}|}$ different values for l. So the search for optimal l at time ${}^4t$ is exponential in $|\mathcal{Y}||\mathcal{M}|$. Similar complexity results hold for ${}^3t$ and ${}^2t$ while at ${}^1t$, c can take uncountably many values. These features makes it challenging to obtain a numerical solution of (44).

The computational difficulty lies in the off-line solution of (44). Once these nested optimality equations have been solved, the on-line implementation of the solution is straightforward.

VI. Discussion

We believe there are two conceptual approaches to solve decentralized multi-agent optimization problems. The first approach is as follows. Arbitrarily fix the decision strategies of all agents. Now pick one agent, say i, and determine its optimal strategy *assuming that the strategy of all other agents is fixed*. Agent i will use this "optimal" strategy in the future. Now pick another agent, say j, $j \neq i$, and determine its optimal strategy assuming that the strategy of all other agents is fixed. Agent j will use this "optimal" strategy in the future. Continue in this manner by cyclically changing the strategies of all agents one-by-one; if the process converges then unilateral deviation by any agent does not improve the system performance. This algorithm is called *orthogonal search*[2] [43] and if it converges it can only guarantee P.B.P.O. (person by person optimal) solutions [36]. However, for team problems P.B.P.O. solutions are not always satisfactory. We want to determine globally optimal strategies, so we do not follow the above described approach to design decision strategies for all agents.

The second approach to solve decentralized multi-agent problems is by considering the problem from the designer's viewpoint. The designer knows the system model and the statistics of the primitive random variables but does not know the observations of any agent. The designer is concerned with determining optimal decision rules for all agents. The state of the system is appropriately defined so that it is sufficient to keep track of the input-output mappings of all agents. The decisions or control actions of the designer are the decision rules ${}^i\varphi_t$ of the original problem. The designer does not take any observations, but remembers all the past decisions. So at time ${}^it$, he knows the system model, the statistics of the primitive random variables, and his past control actions ${}^i\varphi^{t-1}$. The optimization problem at the designer is conceptually equivalent to a POMDP (partially observable Markov decision problem). For POMDP the information state is given by the conditional probability density of the state of the system, conditioned on all the past observations and control actions of the controller (see [2, Chapter 6]). For Problem 1 at time ${}^it$, the random vector $(X_t, {}^iR_t, {}^iB_t)$ can be considered as the system state for the above described alternate formulation. This is because $(X_t, {}^iB_t)$ is a state for input-output mapping[3] at the encoder (this follows from the results of Lemma 1 and Theorem 1) and ${}^iR_t$ is a state for input-output mapping at the receiver. Thus $(X_t, {}^iR_t, {}^iB_t)$ is a state for input-output mapping of the system. However, it is not a state for optimization because the designer does not observe this state. As mentioned above, the optimization problem at the designer is conceptually equivalent to a POMDP. So, a notion that is equivalent to conditional probability conditioned on the controller's observations and control action should be an information state for optimization for the problem at the designer. For this reason in Definition 4, we define the information state at time ${}^it$ as

$$
{}^i\pi_t = \Pr\left(X_t, {}^iR_t, {}^iB_t \,\middle|\, {}^i\varphi^{t-1}\right) \tag{47}
$$

which is the "conditional probability density" of the "state" conditioned on the all the past observations and "control

[2]Fictitious play [42], which is used in game theory to find Nash equilibrium, is a variation of orthogonal search.

[3]See [45] for different notions of state.

actions" of the designer. Technically ${}^i\pi_t$ is not a conditional probability measure, rather it is a unconditional probability measure. But this fact is immaterial for the methodology to solve the problem. The result of Lemma 3 shows that ${}^i\pi_t$ is indeed an information state for optimization which leads to the sequential decomposition given by Theorem 3.

VII. Some Possible Extensions

We have presented the real-time communication problem for the case of zero-delay, first-order Markov source, memoryless forward and backward channels, and finite time horizon. None of these assumptions are crucial; they were made for the ease of exposition. We will briefly describe how to proceed when these assumptions are relaxed. Due to lack of space, we do not present the details of how to proceed in the general case. The details are very similar to the extensions presented in [24] for the case of real-time communication problem with no feedback.

Suppose the distortion measure is not zero-delay but tolerates a fixed-finite delay d. So, for the first d time steps the receiver does not generate the variables $\hat{X}_1, \dots \hat{X}_d$. From $d+1$ onwards, at time t the receiver generates an estimate $\hat{X}_t$ of the source output X_{t-d} at time $t-d$, and incurs an instantaneous cost of $\rho(X_{t-d}, \hat{X}_t)$. This problem can be converted into a problem equivalent to Problem 1 by using the *sliding window repackaging* of the source, similar to what is done in [16], [23], [24]. A similar approach can be used when the source is k^{th} order Markov. A sliding window repackaging converts the source to a first-order Markov source and the results of this paper can be used.

If the forward and the backward channels have memory, the encoder and the decoder need to keep track of the state of the channels. Suppose ${}^iS_t$ and ${}^i\tilde{S}_t$ denote the state of the forward and the backward channel, respectively, at time ${}^it$. Modify the definitions of beliefs as

$$
\begin{aligned}
{}^iB_t &= \Pr\left({}^iR_t, {}^iS_t, {}^i\tilde{S}_t \,\middle|\, {}^i\mathfrak{E}_t\right),\\
{}^iA_t &= \Pr\left({}^iE_t, {}^iS_t, {}^i\tilde{S}_t \,\middle|\, {}^i\mathfrak{R}_t\right),
\end{aligned}
$$

for $i = 1, \dots, 4$, $t = 1, \dots, T$. Then the results of Section III and V can be proved using these modified beliefs. The proof follows along the lines of [24, Section 6].

The formulation of Problem 1 can be extended to infinite horizon, both for the total expected discounted distortion criterion and the average distortion per unit time criterion. Such an extension will result in a corresponding extensions of Problem 2 to infinite horizon, which are standard centralized infinite horizon problems and can be solved by usual fixed point techniques. If the system is "well behaved", the meta-design will be stationary. But the control actions of the deterministic optimization problem, which correspond to the decision rules of Problem 1, will vary with time. So, optimal infinite horizon strategies will be time varying [30], [31].

VIII. Conclusion

We have presented a systematic methodology for designing an optimal communication strategy for real-time communication systems with noisy feedback. Optimal communication strategies can be determined by solving the nested optimality equations of Theorem 3. Note that these are not typical dynamic programming equations as each step is a functional optimization problem. Hence, although the systematic methodology presented here exponentially simplifies the complexity of finding an optimal design as compared to a brute force approach, solving the resultant nested optimality equations is a formidable computational task. We hope that this problem will motivate researchers to investigate computational methods for decentralized optimization problems similar to Problem 1.

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Appendix A
Relation Between the Beliefs

Proof of Lemma 1: We prove the four results separately.

1) Recall that ${}^1e_t := ({}^4e_{t-1}, x_t)$, ${}^1r_t := m_{t-1}$, and ${}^1\phi^{t-1} := ({}^4\phi^{t-2}, l_{t-1})$. Consider any ${}^1e_t$ and ${}^1r_t$, then

$$
\begin{aligned}
{}^1b_t({}^1r_t) &:= \Pr\left({}^1r_t \,\middle|\, {}^1e_t; {}^1\phi^{t-1}\right)\\
&= \Pr\left(m_{t-1} \,\middle|\, {}^4e_{t-1}, x_t; {}^4\phi^{t-2}, l_{t-1}\right)\\
&= \sum_{\substack{y_{t-1}\in\mathcal{Y}\\ m_{t-2}\in\mathcal{M}}} \Pr\left(y_{t-1}, m_{t-1}, m_{t-2} \,\middle|\, {}^4e_{t-1}, x_t; {}^4\phi^{t-2}, l_{t-1}\right)\\
&= \sum_{\substack{y_{t-1}\in\mathcal{Y}\\ m_{t-2}\in\mathcal{M}}} \Pr\left(m_{t-1} \,\middle|\, y_{t-1}, m_{t-2}, {}^4e_{t-1}, x_t; {}^4\phi^{t-2}, l_{t-1}\right)\\
&\qquad \times \Pr\left(y_{t-1}, m_{t-2} \,\middle|\, {}^4e_{t-1}, x_t; {}^4\phi^{t-2}, l_{t-1}\right)\\
&\overset{(a)}{=} \sum_{\substack{y_{t-1}\in\mathcal{Y}\\ m_{t-2}\in\mathcal{M}}} \mathbb{I}\left[m_{t-1} = l_{t-1}(y_{t-1}, m_{t-2})\right]\\
&\qquad \times \Pr\left(y_{t-1}, m_{t-2} \,\middle|\, {}^4e_t; {}^4\phi^{t-2}\right)\\
&= \sum_{\substack{y_{t-1}\in\mathcal{Y}\\ m_{t-2}\in\mathcal{M}}} \mathbb{I}\left[m_{t-1} = l_{t-1}(y_{t-1}, m_{t-2})\right]\\
&\qquad \times {}^4b_{t-1}(y_{t-1}, m_{t-2})\\
&=: {}^1F({}^4b_{t-1}, l_{t-1})(m_{t-1}) = {}^1F({}^4b_{t-1}, l_{t-1})({}^1r_t),
\end{aligned} \tag{48}
$$

where (a) follows from the sequential order in which the system variables are generated.

2) Recall that ${}^2e_t := ({}^1e_t, z_t)$, ${}^2r_t := (y_t, m_{t-1})$, and ${}^2\phi^{t-1} := ({}^1\phi^{t-1}, c_t)$. Consider any ${}^2e_t$ and ${}^2r_t$, then

$$
\begin{aligned}
{}^2b_t({}^2r_t) &:= \Pr\left({}^2r_t \,\middle|\, {}^2e_t; {}^2\phi^{t-1}\right)\\
&= \Pr\left(y_t, m_{t-1} \,\middle|\, {}^1e_t, z_t; {}^1\phi^{t-1}, c_t\right)\\
&= \Pr\left(y_t \,\middle|\, {}^1e_t, z_t, m_{t-1}; {}^1\phi^{t-1}, c_t\right)\\
&\quad \times \Pr\left(m_{t-1} \,\middle|\, {}^1e_t, z_t; {}^1\phi^{t-1}, c_t\right)\\
&\overset{(b)}{=} P_N\left(n_t \in \mathcal{N} : y_t = h(z_t, n_t)\right)\\
&\quad \times \Pr\left(m_{t-1} \,\middle|\, {}^1e_t; {}^1\phi^{t-1}\right)\\
&= P_N\left(n_t \in \mathcal{N} : y_t = h(z_t, n_t)\right) {}^1b_t(m_{t-1})\\
&=: {}^2F({}^1b_t, z_t)(y_t, m_{t-1}) = {}^2F({}^1b_t, z_t)({}^2r_t),
\end{aligned} \tag{49}
$$

where (b) follows from the sequential order in which the system variables are generated.

3) Recall that ${}^3e_t := {}^2e_t$, ${}^3r_t := (y_t, m_{t-1})$, and ${}^3\phi^{t-1} := ({}^2\phi^{t-1}, g_t)$. Consider any ${}^3e_t$ and ${}^3r_t$, then

$$\begin{aligned} {}^3b_t({}^3r_t) &:= \Pr\left({}^3r_t \,\middle|\, {}^3e_t; {}^3\phi^{t-1}\right) \\ &= \Pr\left(y_t, m_{t-1} \,\middle|\, {}^2e_t; {}^2\phi^{t-1}, g_t\right) \\ &\overset{(c)}{=} \Pr\left(y_t, m_{t-1} \,\middle|\, {}^2e_t; {}^2\phi^{t-1}\right) \\ &=: {}^2b_t(y_t, m_{t-1}) = {}^2b_t({}^3r_t), \end{aligned} \tag{50}$$

where (c) follows from the sequential order in which the system variables are generated.

4) Recall that ${}^4e_t := ({}^3e_t, \tilde{y}_t)$, ${}^4r_t := (y_t, m_{t-1})$, and ${}^4\phi^{t-1} := ({}^3\phi^{t-1}, \tilde{c}_t)$. Consider any ${}^4e_t$ and ${}^4r_t$, then

$$\begin{aligned} {}^4b_t({}^4r_t) &:= \Pr\left({}^4r_t \,\middle|\, {}^4e_t; {}^4\phi^{t-1}\right) \\ &= \Pr\left(y_t, m_{t-1} \,\middle|\, {}^3e_t, \tilde{y}_t; {}^3\phi^{t-1}, \tilde{c}_t\right) \\ &= \frac{\Pr\left(y_t, m_{t-1}, \tilde{y}_t \,\middle|\, {}^3e_t; {}^3\phi^{t-1}, \tilde{c}_t\right)}{\Pr\left(\tilde{y}_t \,\middle|\, {}^3e_t; {}^3\phi^{t-1}, \tilde{c}_t\right)}. \end{aligned} \tag{51}$$

Now consider

$$\begin{aligned} &\Pr\left(\tilde{y}_t, y_t, m_{t-1} \,\middle|\, {}^3e_t; {}^3\phi^{t-1}, \tilde{c}_t\right) \\ &\quad= \sum_{\tilde{z} \in \tilde{\mathcal{Z}}} \Pr\left(\tilde{y}_t, z_t, y_t, m_{t-1} \,\middle|\, {}^3e_t; {}^3\phi^{t-1}, \tilde{c}_t\right) \\ &\quad= \sum_{\tilde{z} \in \tilde{\mathcal{Z}}} \Pr\left(\tilde{y}_t \,\middle|\, \tilde{z}_t, y_t, m_{t-1}, {}^3e_t; {}^3\phi^{t-1}, \tilde{c}_t\right) \\ &\qquad\qquad \times \Pr\left(\tilde{z}_t \,\middle|\, y_t, m_{t-1}, {}^3e_t; {}^3\phi^{t-1}, \tilde{c}_t\right) \\ &\qquad\qquad \times \Pr\left(y_t, m_{t-1} \,\middle|\, {}^3e_t; {}^3\phi^{t-1}, \tilde{c}_t\right) \\ &\quad\overset{(d)}{=} \sum_{\tilde{z} \in \tilde{\mathcal{Z}}} P_{\tilde{N}}\left(\tilde{n}_t \in \tilde{\mathcal{N}} : \tilde{y}_t = \tilde{h}(\tilde{z}_t, \tilde{n}_t)\right) \\ &\qquad\qquad \times \mathbb{I}\left[\tilde{z}_t = \tilde{c}_t(y_t, m_{t-1})\right] \\ &\qquad\qquad \times \Pr\left(y_t, m_{t-1} \,\middle|\, {}^3e_t; {}^3\phi^{t-1}\right), \\ &\quad= \sum_{\tilde{z} \in \tilde{\mathcal{Z}}} P_{\tilde{N}}\left(\tilde{n}_t \in \tilde{\mathcal{N}} : \tilde{y}_t = \tilde{h}(\tilde{z}_t, \tilde{n}_t)\right) \\ &\qquad\qquad \times \mathbb{I}\left[\tilde{z}_t = \tilde{c}_t(y_t, m_{t-1})\right] \\ &\qquad\qquad \times {}^3b_t(y_t, m_{t-1}), \end{aligned} \tag{52}$$

where (d) follows from the sequential order in which the system variables are generated. Observe that $\Pr(\tilde{y}_t \mid {}^3e_t; {}^3\phi^{t-1}, \tilde{c}_t)$ is the marginal of the LHS of (52). Combine (51) and (52) to obtain

$$\begin{aligned} {}^4b_t({}^4r_t) &=: {}^4F({}^3b_t, \tilde{y}_t, \tilde{c}_t)(y_t, m_{t-1}) \\ &= {}^4F({}^3b_t, \tilde{y}_t, \tilde{c}_t)({}^4r_t), \end{aligned} \tag{53}$$

where ${}^4F$ is defined by (51) and (52).

■

Proof of Lemma 2: We prove the five results separately.

1) Recall that ${}^1e_t := (x^t, z^{t-1}, \tilde{y}^{t-1})$, ${}^1R_t := m_{t-1}$, and ${}^1\phi^{t-1} := (c^{t-1}, g^{t-1}, \tilde{c}^{t-1}, l^{t-1})$. Consider any ${}^1e_t$ and ${}^1r_t$, then

$$\begin{aligned} {}^1a_t({}^1e_t) &:= \Pr\left({}^1e_t \,\middle|\, {}^1R_t; {}^1\phi^{t-1}\right) \\ &= \Pr\left(x^t, z^{t-1}, \tilde{y}^{t-1} \,\middle|\, m_{t-1}; {}^1\phi^{t-1}\right) \\ &= \frac{1}{\Pr\left(m_{t-1} \,\middle|\, {}^1\phi^{t-1}\right)} \\ &\quad \times \sum_{\substack{y^{t-1} \in \mathcal{Y}^{t-1} \\ m^{t-2} \in \mathcal{M}^{t-2}}} \Pr\left(x^t, z^{t-1}, y^{t-1}, \tilde{y}^{t-1}, m^{t-1} \,\middle|\, {}^1\phi^{t-1}\right). \end{aligned} \tag{54}$$

Now consider

$$\begin{aligned} &\Pr\left(x^t, z^{t-1}, y^{t-1}, \tilde{y}^{t-1}, m^{t-1} \,\middle|\, {}^1\phi^{t-1}\right) \\ &\quad= P_{x_1}(x_1)\, \mathbb{I}\left[z_1 = c_1(x_1)\right] \\ &\qquad \times P_N\left(n_1 \in \mathcal{N} : y_1 = h(z_1, n_1)\right) \mathbb{I}\left[\tilde{z}_1 = \tilde{c}(y_1)\right] \\ &\qquad \times P_{\tilde{N}}\left(\tilde{n}_1 \in \tilde{\mathcal{N}} : \tilde{y}_1 = \tilde{h}(\tilde{z}_1, \tilde{n}_1)\right) \mathbb{I}\left[m_1 = l_1(y_1)\right] \\ &\qquad \times \prod_{i=2}^{t-1} \Big\{ P_{X_i|X_{i-1}}(x_i \mid x_{i-1}) \\ &\qquad\qquad \times \mathbb{I}\left[z_i = c_i(x^i, z^{i-1}, \tilde{y}^{i-1})\right] \\ &\qquad\qquad \times P_N\left(n_i \in \mathcal{N} : y_i = h(z_i, n_i)\right) \\ &\qquad\qquad \times \mathbb{I}\left[\tilde{z}_i = \tilde{c}(y_i, m_{i-1})\right] \\ &\qquad\qquad \times P_{\tilde{N}}\left(\tilde{n}_i \in \tilde{\mathcal{N}} : \tilde{y}_i = \tilde{h}(\tilde{z}_i, \tilde{n}_i)\right) \\ &\qquad\qquad \times \mathbb{I}\left[m_i = l_i(y_i, m_{i-1})\right] \Big\} \\ &\qquad \times P_{X_t|X_{t-1}}(x_t \mid x_{t-1}). \end{aligned} \tag{55}$$

Observe that $\Pr(m_{t-1} \mid {}^1\phi^{t-1})$ is the marginal of the LHS of (55). Combine (54) and (55) to obtain

$$\begin{aligned} &{}^1a_t({}^1e_t) \\ &\quad=: {}^1K_t({}^1a_1, m_{t-1}, c^{t-1}, \tilde{c}^{t-1}, l^{t-1})(x^t, z^{t-1}, \tilde{y}^{t-1}) \\ &\quad= {}^1K_t({}^1a_1, m_{t-1}, c^{t-1}, \tilde{c}^{t-1}, l^{t-1})({}^1e_t), \end{aligned} \tag{56}$$

where ${}^1K_t(\cdot)$ is given by (54) and (55).

2) Recall that ${}^2e_t := ({}^1e_t, z_t)$, ${}^2R_t := (y_t, m_{t-1})$, and ${}^2\phi^{t-1} := ({}^1\phi^{t-1}, c_t)$. Consider any ${}^2e_t$ and ${}^2r_t$, then

$$\begin{aligned} {}^2a_t({}^2e_t) &:= \Pr\left({}^2e_t \,\middle|\, {}^2R_t; {}^2\phi^{t-1}\right) \\ &= \Pr\left({}^1e_t, z_t \,\middle|\, y_t, m_{t-1}; {}^1\phi^{t-1}, c_t\right) \\ &= \frac{\Pr\left({}^1e_t, z_t, y_t \,\middle|\, m_{t-1}; {}^1\phi^{t-1}, c_t\right)}{\Pr\left(y_t \,\middle|\, m_{t-1}; {}^1\phi^{t-1}, c_t\right)}. \end{aligned} \tag{57}$$

Now consider

$$\begin{aligned} &\Pr\left({}^1e_t, z_t, y_t \,\middle|\, m_{t-1}; {}^1\phi^{t-1}, c_t\right) \\ &\quad= \Pr\left(y_t \,\middle|\, z_t, {}^1e_t, m_{t-1}; {}^1\phi^{t-1}, c_t\right) \\ &\qquad \times \Pr\left(z_t \,\middle|\, {}^1e_t, m_{t-1}; {}^1\phi^{t-1}, c_t\right) \\ &\qquad \times \Pr\left({}^1e_t \,\middle|\, m_{t-1}; {}^1\phi^{t-1}, c_t\right) \\ &\quad\overset{(a)}{=} P_N\left(n_t \in \mathcal{N} : y_t = h(z_t, n_t)\right) \\ &\qquad \times \mathbb{I}\left[z_t = c_t({}^1e_t)\right] \\ &\qquad \times \Pr\left({}^1e_t \,\middle|\, m_{t-1}; {}^1\phi^{t-1}\right) \\ &\quad= P_N\left(n_t \in \mathcal{N} : y_t = h(z_t, n_t)\right) \\ &\qquad \times \mathbb{I}\left[z_t = c_t({}^1e_t)\right] \\ &\qquad \times {}^1a_t({}^1e_t), \end{aligned} \tag{58}$$

where (a) follows from the sequential order in which the system variables are generated. Observe that $\Pr(m_{t-1} \mid {}^1\phi^{t-1}, c_t)$ is the marginal of the LHS of (58). Combine (57) and (58) to obtain

$$
\begin{aligned}
{}^2a_t({}^2e_t) &=: {}^2K({}^1a_t, y_t, c_t)({}^1e_t, z_t) \\
&= {}^2K({}^1a_t, y_t, c_t)({}^2e_t), \quad (59)
\end{aligned}
$$

where ${}^2K(\cdot)$ is given by (57) and (58).

3) Recall that ${}^2e_t := ({}^4e_{t-1}, x_t, z_t)$. Consider any x_t and ${}^2r_t$, then

$$
\begin{aligned}
\hat{a}_t(x_t) &:= \Pr\left(x_t \,\middle|\, {}^2R_t; {}^2\phi^{t-1}\right) \\
&= \sum_{\substack{z_t \in \mathcal{Z} \\ {}^4e_{t-1} \in {}^4\mathcal{E}_{t-1}}} \Pr\left(x_t, z_t, {}^4e_{t-1} \,\middle|\, {}^2R_t; {}^2\phi^{t-1}\right) \\
&= \sum_{\substack{z_t \in \mathcal{Z} \\ {}^4e_{t-1} \in {}^4\rceil_{t-1}}} {}^2a_t(x_t, z_t, {}^4e_{t-1}) =: \hat{K}({}^2a_t)(x_t).
\end{aligned}
$$

4) Recall that ${}^3e_t := {}^2e_t$, ${}^3R_t := (y_t, m_{t-1})$, and ${}^3\phi^{t-1} := ({}^2\phi^{t-1}, g_t)$. Consider any ${}^3e_t$ and ${}^3r_t$, then

$$
\begin{aligned}
{}^3a_t({}^3e_t) &:= \Pr\left({}^3e_t \,\middle|\, {}^3R_t; {}^3\phi^{t-1}\right) \\
&= \Pr\left({}^2e_t \,\middle|\, y_t, m_{t-1}; {}^2\phi^{t-1}, g_t\right) \\
&\overset{(b)}{=} \Pr\left({}^2e_t \,\middle|\, y_t, m_{t-1}; {}^2\phi^{t-1}\right) \\
&=: {}^2a_t({}^2e_t) = {}^2a_t({}^3e_t). \quad (60)
\end{aligned}
$$

where (b) follows from the sequential order in which the system variables are generated.

5) Recall that ${}^4e_t := ({}^3e_t, \tilde{y}_t)$, ${}^4R_t := (y_t, m_{t-1})$, and ${}^4\phi^{t-1} := ({}^3\phi^{t-1}, \tilde{c}_t)$. Consider any ${}^4e_t$ and ${}^4r_t$, then

$$
\begin{aligned}
{}^4a_t({}^4e_t) &:= \Pr\left({}^4e_t \,\middle|\, {}^4R_t; {}^4\phi^{t-1}\right) \\
&= \Pr\left({}^3e_t, \tilde{y}_t \,\middle|\, y_t, m_{t-1}; {}^3\phi^{t-1}, \tilde{c}_t\right) \\
&\overset{(c)}{=} \Pr\left({}^3e_t, \tilde{y}_t \,\middle|\, y_t, m_{t-1}, \tilde{z}_t; {}^3\phi^{t-1}, \tilde{c}_t\right) \\
&= \Pr\left(\tilde{y}_t \,\middle|\, {}^3e_t, y_t, m_{t-1}, \tilde{z}_t; {}^3\phi^{t-1}, \tilde{c}_t\right) \\
&\quad \times \Pr\left({}^3e_t \,\middle|\, y_t, m_{t-1}, \tilde{z}_t; {}^3\phi^{t-1}, \tilde{c}_t\right) \\
&\overset{(d)}{=} P_{\tilde{N}}\left(\tilde{n}_t \in \tilde{\mathcal{N}} : \tilde{y}_t = \tilde{h}(\tilde{z}_t, \tilde{n}_t)\right) \\
&\quad \times \Pr\left({}^3e_t \,\middle|\, y_t, m_{t-1}; {}^3\phi^{t-1}\right) \\
&= P_{\tilde{N}}\left(\tilde{n}_t \in \tilde{\mathcal{N}} : \tilde{y}_t = \tilde{h}(\tilde{z}_t, \tilde{n}_t)\right) {}^3a_t({}^3e_t) \\
&=: {}^4K({}^3a_t, \tilde{z}_t)({}^3e_t, \tilde{y}_t) = {}^4K({}^3a_t, \tilde{z}_t)({}^4e_t), \quad (61)
\end{aligned}
$$

where (c) follows from (4) and (d) follows from the sequential order in which the system variables are generated. ■

APPENDIX B
RELATION BETWEEN INFORMATION STATES

Proof of Lemma 3: We prove the four results separately.

1) Recall that ${}^2r_t := (y_t, {}^1r_t)$, ${}^2\phi^{t-1} := ({}^1\phi^{t-1}, c_t)$, $z_t = c_t(x_t, {}^1r_t)$, and ${}^2b_t = {}^2F({}^1b_t, z_t)$. Consider a component of ${}^2\pi_t$,

$$
\begin{aligned}
{}^2\pi_t &:= \Pr\left(x_t, {}^2r_t, {}^2b_t \,\middle|\, {}^2\phi^{t-1}\right) \\
&= \Pr\left(x_t, y_t, {}^1r_t, {}^2b_t \,\middle|\, {}^2\phi^{t-1}\right) \\
&= \int_{{}^1b_t \in {}^1\mathcal{B}} \sum_{z_t \in \mathcal{Z}} \Pr\left(x_t, y_t, z_t, {}^1r_t, {}^1b_t, {}^2b_t \,\middle|\, {}^2\phi^{t-1}\right) d\,{}^1b_t \\
&= \int_{{}^1b_t \in {}^1\mathcal{B}} \sum_{z_t \in \mathcal{Z}} \Pr\left({}^2b_t \,\middle|\, x_t, y_t, z_t, {}^1r_t, {}^1b_t; {}^1\phi^{t-1}, c_t\right) \\
&\qquad \times \Pr\left(y_t \,\middle|\, x_t, z_t, {}^1r_t, {}^1b_t; {}^1\phi^{t-1}, c_t\right) \\
&\qquad \times \Pr\left(z_t \,\middle|\, x_t, {}^1r_t, {}^1b_t; {}^1\phi^{t-1}, c_t\right) \\
&\qquad \times \Pr\left(x_t, {}^1r_t, {}^1b_t \,\middle|\, {}^1\phi^{t-1}, c_t\right) d\,{}^1b_t \\
&= \int_{{}^1b_t \in {}^1\mathcal{B}} \sum_{z_t \in \mathcal{Z}} \mathbb{I}\left[{}^2b_t = {}^2F({}^1b_t, z_t)\right] \\
&\qquad \times P_N\left(n_t \in \mathcal{N} : y_t = h(z_t, n_t)\right) \\
&\qquad \times \mathbb{I}\left[z_t = c_t(x_t, {}^1b_t)\right] \\
&\qquad \times \Pr\left(x_t, {}^1r_t, {}^1b_t \,\middle|\, {}^1\phi^{t-1}\right) d\,{}^1b_t \\
&= \int_{{}^1b_t \in {}^1\mathcal{B}} \sum_{z_t \in \mathcal{Z}} \mathbb{I}\left[{}^2b_t = {}^2F({}^1b_t, z_t)\right] \\
&\qquad \times P_N\left(n_t \in \mathcal{N} : y_t = h(z_t, n_t)\right) \\
&\qquad \times \mathbb{I}\left[z_t = c_t(x_t, {}^1b_t)\right] \\
&\qquad \times {}^1\pi_t(x_t, {}^1r_t, {}^1b_t) d\,{}^1b_t \\
&=: {}^1Q(c_t)\, {}^1\pi_t \quad (62)
\end{aligned}
$$

2) Recall that ${}^3r_t := {}^2r_t$, ${}^3b_t := {}^2b_t$, and ${}^3\phi^{t-1} := ({}^2\phi^{t-1}, g_t)$. Consider a component of ${}^3\pi_t$,

$$
\begin{aligned}
{}^3\pi_t(x_t, {}^3r_t, {}^3b_t) &:= \Pr\left(x_t, {}^3r_t, {}^3b_t \,\middle|\, {}^3\phi^{t-1}\right) \\
&= \Pr\left(x_t, {}^2r_t, {}^2b_t \,\middle|\, {}^2\phi^{t-1}, g_t\right) \\
&\overset{(a)}{=} \Pr\left(x_t, {}^2r_t, {}^2b_t \,\middle|\, {}^2\phi^{t-1}\right) \\
&=: {}^2\pi_t(x_t, {}^2r_t, {}^2b_t), \quad (63)
\end{aligned}
$$

where (a) follows from the sequential order in which the system variables are generated.

3) Recall that ${}^4r_t := {}^3r_t$, ${}^4\phi^{t-1} := ({}^3\phi^{t-1}, \tilde{c}_t)$, $\tilde{z}_t = \tilde{c}_t({}^3r_t)$, and ${}^4b_t = {}^4F({}^3b_t, \tilde{y}_t, \tilde{c}_t)$. Consider a component of ${}^4\pi_t$,

$$
\begin{aligned}
&{}^4\pi_t(x_t, {}^4r_t, {}^4b_t) := \Pr\left(x_t, {}^4r_t, {}^4b_t \,\middle|\, {}^4\phi^{t-1}\right) \\
&= \Pr\left(x_t, {}^3r_t, {}^4b_t \,\middle|\, {}^3\phi^{t-1}, \tilde{c}_t\right) \\
&= \int_{{}^3b_t \in {}^3\mathcal{B}} \sum_{\substack{\tilde{z}_t \in \tilde{\mathcal{Z}} \\ \tilde{y}_t \in \tilde{\mathcal{Y}}}} \Pr\left(x_t, \tilde{z}_t, \tilde{y}_t, {}^3r_t, {}^3b_t, {}^4b_t \,\middle|\, {}^3\phi^{t-1}, \tilde{c}_t\right) d\,{}^3b_t \\
&= \int_{{}^3b_t \in {}^3\mathcal{B}} \sum_{\substack{\tilde{z}_t \in \tilde{\mathcal{Z}} \\ \tilde{y}_t \in \tilde{\mathcal{Y}}}} \Pr\left({}^4b_t \,\middle|\, x_t, \tilde{z}_t, \tilde{y}_t, {}^3r_t, {}^3b_t; {}^3\phi^{t-1}, \tilde{c}_t\right) \\
&\qquad \times \Pr\left(\tilde{y}_t \,\middle|\, x_t, \tilde{z}_t, {}^3r_t, {}^3b_t; {}^3\phi^{t-1}, \tilde{c}_t\right) \\
&\qquad \times \Pr\left(\tilde{z}_t \,\middle|\, x_t, {}^3r_t, {}^3b_t; {}^3\phi^{t-1}, \tilde{c}_t\right) \\
&\qquad \times \Pr\left(x_t, {}^3r_t, {}^3b_t \,\middle|\, {}^3\phi^{t-1}, \tilde{c}_t\right) d\,{}^3b_t \\
&\overset{(b)}{=} \int_{{}^3b_t \in {}^3\mathcal{B}} \sum_{\substack{\tilde{z}_t \in \tilde{\mathcal{Z}} \\ \tilde{y}_t \in \tilde{\mathcal{Y}}}} \mathbb{I}\left[{}^4b_t = {}^4F({}^3b_t, \tilde{y}_t, \tilde{c}_t)\right] \\
&\qquad \times P_{\tilde{N}}\left(\tilde{n}_t \in \tilde{\mathcal{N}} : \tilde{y}_t = \tilde{h}(\tilde{z}_t, \tilde{n}_t)\right) \\
&\qquad \times \mathbb{I}\left[\tilde{z}_t = \tilde{c}_t({}^3r_t)\right] \\
&\qquad \times \Pr\left(x_t, {}^3r_t, {}^3b_t \,\middle|\, {}^3\phi^{t-1}\right) d\,{}^3b_t \\
&= \int_{{}^3b_t \in {}^3\mathcal{B}} \sum_{\substack{\tilde{z}_t \in \tilde{\mathcal{Z}} \\ \tilde{y}_t \in \tilde{\mathcal{Y}}}} \mathbb{I}\left[{}^4b_t = {}^4F({}^3b_t, \tilde{y}_t, \tilde{c}_t)\right] \\
&\qquad \times P_{\tilde{N}}\left(\tilde{n}_t \in \tilde{\mathcal{N}} : \tilde{y}_t = \tilde{h}(\tilde{z}_t, \tilde{n}_t)\right) \\
&\qquad \times \mathbb{I}\left[\tilde{z}_t = \tilde{c}_t({}^3r_t)\right] \\
&\qquad \times {}^3\pi_t(x_t, {}^3r_t, {}^3b_t) d\,{}^3b_t \\
&=: {}^3Q(\tilde{c}_t)\, {}^3\pi_t, \quad (64)
\end{aligned}
$$

where (b) follows from the sequential order in which the system variables are generated.

4) Recall that ${}^1\phi^t := ({}^4\phi^{t-1}, l_t)$, ${}^1r_{t+1} = l_t({}^4r_t)$, and ${}^1b_{t+1} = {}^1F({}^4b_t, l_t)$. Consider a component of ${}^1\pi_{t+1}$,

$$
\begin{aligned}
&{}^1\pi_{t+1}(x_{t+1}, {}^1r_{t+1}, {}^1b_{t+1}) \\
&:= \Pr\left(x_{t+1}, {}^1r_{t+1}, {}^1b_{t+1} \,\middle|\, {}^1\phi^t\right) \\
&= \int_{{}^4b_t \in {}^4\mathcal{B}} \sum_{\substack{x_t \in \mathcal{X} \\ {}^4r_t \in {}^4\mathcal{R}_t}} \Pr\left(x_{t+1}, x_t, {}^4r_t, {}^1r_{t+1}, {}^4b_t, {}^1b_{t+1} \,\middle|\, {}^1\phi^t\right) d\,{}^4b_t \\
&= \int_{{}^4b_t \in {}^4\mathcal{B}} \sum_{\substack{x_t \in \mathcal{X} \\ {}^4r_t \in {}^4\mathcal{R}_t}} \Pr\left(x_t, {}^4r_t, {}^4b_t \,\middle|\, {}^4\phi^{t-1}, l_t\right) \\
&\qquad \times \Pr\left({}^1r_{t+1} \,\middle|\, x_t, {}^4r_t, {}^4b_t; {}^4\phi^{t-1}, l_t\right) \\
&\qquad \times \Pr\left(x_{t+1} \,\middle|\, x_t, {}^4r_t, {}^1r_{t+1}, {}^4b_t; {}^4\phi^{t-1}, l_t\right) \\
&\qquad \times \Pr\left({}^1b_{t+1} \,\middle|\, x_{t+1}, x_t, {}^4r_t, {}^1r_{t+1}, {}^4b_t; {}^4\phi^{t-1}, l_t\right) d\,{}^4b_t \\
&\overset{(c)}{=} \int_{{}^4b_t \in {}^4\mathcal{B}} \sum_{\substack{x_t \in \mathcal{X} \\ {}^4r_t \in {}^4\mathcal{R}_t}} \mathbb{I}\left[{}^1b_{t+1} = {}^1F({}^4b_t, l_t)\right] \\
&\qquad \times P_{x_{t+1}|x_t}(x_{t+1} \mid x_t) \\
&\qquad \times \mathbb{I}\left[{}^1r_{t+1} = l_t({}^4r_t)\right] \\
&\qquad \times \Pr\left(x_t, {}^4r_t, {}^4b_t \,\middle|\, {}^4\phi^{t-1}\right) d\,{}^4b_t \\
&= \int_{{}^4b_t \in {}^4\mathcal{B}} \sum_{\substack{x_t \in \mathcal{X} \\ {}^4r_t \in {}^4\mathcal{R}_t}} \mathbb{I}\left[{}^1b_{t+1} = {}^1F({}^4b_t, l_t)\right] \\
&\qquad \times P_{x_{t+1}|x_t}(x_{t+1} \mid x_t) \\
&\qquad \times \mathbb{I}\left[{}^1r_{t+1} = l_t({}^4r_t)\right] \\
&\qquad \times {}^4\pi_t(x_t, {}^4r_t, {}^4b_t) d\,{}^4b_t \\
&=: {}^4Q(l_t)\,{}^4\pi_t, \qquad (65)
\end{aligned}
$$

where (c) follows from the sequential order in which the system variables are generated.

5) Recall that ${}^2r_t := (y_t, m_{t-1})$ and $\hat{x}_t = g_t({}^2r_t)$. Consider

$$
\begin{aligned}
&\mathbb{E}\left\{\rho(x_t, \hat{x}_t) \,\middle|\, {}^2\phi^{t-1}, g_t\right\} \\
&\quad = \sum_{\substack{x_t \in \mathcal{X} \\ {}^2r_t \in {}^2\mathcal{R}_t}} \Pr\left(x_t, {}^2r_t \,\middle|\, {}^2\phi^{t-1}, g_t\right) \rho\big(x_t, g_t({}^2r_t)\big) \\
&\quad = \sum_{\substack{x_t \in \mathcal{X} \\ {}^2r_t \in {}^2\mathcal{R}_t}} \Pr\left(x_t, {}^2r_t \,\middle|\, {}^2\phi^{t-1}\right) \rho\big(x_t, g_t({}^2r_t)\big) \\
&\quad = \sum_{\substack{x_t \in \mathcal{X} \\ {}^2r_t \in {}^2\mathcal{R}_t}} {}^2\pi_t(x_t, {}^2r_t) \rho\big(x_t, g_t({}^2r_t)\big) =: \hat{\rho}({}^2\pi_t, g_t),
\end{aligned} \tag{66}
$$

where ${}^2\pi_t(x_t, {}^2r_t)$ is the marginal of ${}^2\pi_t(x_t, {}^2r_t, {}^2b_t)$.

The linearity of ${}^1Q(\cdot)$, ${}^3Q(\cdot)$, and ${}^4Q(\cdot)$ in the corresponding ${}^i\pi_t$, and the concavity of $\hat{\rho}$ in ${}^2\pi$ follow immediately from their definition. ∎

Appendix C
Concavity of Value Functions

Proof of Theorem 4: Recall that ${}^iQ(\cdot)$, $i = 1, 3, 4$, is a linear transformations of the corresponding ${}^i\pi$ and $\hat{\rho}$ is a concave function in ${}^2\pi$. We will prove concavity of the value function by backward induction. Observe that ${}^1V_{T+1}$ is a concave function of ${}^1\pi$. Now assume that ${}^1V_{t+1}$ is a concave function of ${}^1\pi$. We will show that ${}^iV_t$, $i = 1, \dots, T$ are concave functions of ${}^i\pi$. Define

$$
{}^4W_t({}^4\pi, l) := {}^1V_{t+1}({}^4Q(l)\,{}^4\pi). \tag{67}
$$

As a function of ${}^4\pi$, ${}^4W$ is a composition of a concave function with a linear transformation. Hence ${}^4W$ is concave in ${}^4\pi$. Now,

$$
{}^4V_t({}^4\pi) = \min_{l \in \mathscr{L}} {}^4W({}^4\pi, l). \tag{68}
$$

Since ${}^4W_t$ is concave in ${}^4\pi$, and ${}^4V_t$ is the point-wise minimum of ${}^4W_t$, ${}^4V_t$ is concave in ${}^4\pi$. Similar argument extends to the other three cases. Hence ${}^iV_t$ is concave in ${}^i\pi$, $i = 1, \dots, 4$. Thus, by induction ${}^iV_t$, $i = 1, \dots, 4$, $t = 1, \dots, T$ is concave in ${}^i\pi$. ∎

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