On the Structure of Optimal Real-Time Encoders and Decoders in Noisy Communication

Demosthenis Teneketzis, *Fellow, IEEE*

***Abstract*—The output of a discrete-time Markov source must be encoded into a sequence of discrete variables. The encoded sequence is transmitted through a noisy channel to a receiver that must attempt to reproduce reliably the source sequence. Encoding and decoding must be done in real-time and the distortion measure does not tolerate delays. The structure of real-time encoding and decoding strategies that jointly minimize an average distortion measure over a finite horizon is determined. The results are extended to the real-time broadcast problem and a real-time variation of the Wyner–Ziv problem.**

***Index Terms*—Markov chains, Markov decision theory, real-time decoding, real-time encoding.**

I. Introduction

In a point-to-point communication system the outputs of a discrete-time Markov source are encoded into a sequence of discrete variables. This sequence is transmitted through a noisy channel to a receiver (decoder), which must attempt to reproduce the outputs of the Markov source. Operation is in real-time. That is, the encoding of each source symbol at the transmitter and its decoding at the receiver must be performed without any delay and the distortion measure does not tolerate delays. Similar real-time encoding problems are considered for the broadcast system, [33], and a variation of the Wyner–Ziv problem [34]. The real-time constraint is motivated by controlled informationally decentralized systems (such as networks) where information must be exchanged among various sites of the system in real-time, and decisions, using the communicated information, have to be made in real-time.

Problems with the real-time constraint on information transmission are drastically different from the classical information theory problem for the following reasons. The fundamental results of information theory are asymptotic in nature. They deal with the encoding of long sequences that are asymptotically "typical". Encoding of long sequences introduces long undesirable delays in communication. The information theoretic results available on the trade-off between delay and reliability ([1, Ch. 5]) are asymptotically tight but of limited value for short sequences. Furthermore, channel capacity, which is a key concept in information theory, is inappropriate here because it is an asymptotic concept. As pointed out in [2], channels with the same capacity may behave quite differently under the real-time constraint.

Real-time encoding-decoding problems have received significant attention. Necessary conditions that an optimal digital system with a real-time encoder and decoder must satisfy were presented in [3]. These conditions were applied to pulse code and delta modulation systems. Real-time communication over infinite time spans was investigated in [15] where attention was restricted to myopic encoding rules. The real-time transmission of a memoryless source over a memoryless channel was investigated in [30], [71], where it was shown that memoryless encoders and decoders are optimal.

Causal lossy encoding for memoryless, stationary and binary symmetric first-order Markov sources was investigated in [4]–[6], [45], [68] where optimal causal encoders were determined for memoryless and stationary sources. As pointed out in [4, p. 702], the notion of causality used in [4]–[6], [45], [68] is weaker than the real-time requirement considered in this paper.

The existence and structure of optimal real-time encoding strategies for systems with noiseless (error-free) channels, different types of sources (e.g., Bernoulli processes, Markov processes, sequences of bounded uniformly distributed random variables, etc.) was investigated and discovered in [7]–[15], [50]–[52]. Error exponents for real-time encoding of discrete memoryless sources were derived in [53].

The structure of optimal real-time encoding and decoding strategies for systems with noisy channels, perfect feedback from the output of the channel to the encoder, and various performance criteria was investigated in [2], [18], [19]. Applications of the results developed in [18] appeared in [20], [21].

Bounds on the performance of communication systems with the real-time or finite delay constraint on information transmission were obtained via different methods (e.g., mathematical programming, forward flow of information, and other information theoretic methods including conditional mutual information and the determination of nonanticipatory rate distortion functions) in [22]–[29].

Real-time or finite delay encoding-decoding problems, as well as the sensitivity of reliable communication with respect to delays in transmission and decoding, were investigated in [46]. In [46] a new notion of capacity (called "anytime capacity) that corresponds to a sense of reliable transmission and is different from the Shannon capacity was defined.

The stochastic stability of causal encoding schemes (including adaptive quantization, delta modulation, differential

Manuscript received February 13, 2004; revised November 17, 2005. This research was supported in part by NSF Grant ECS-9979347, NSF Grant CCR-0082784, ONR Grant N00014-03-1-0232, and NSF Grant CCR-0325571. The material in this paper was presented at the Workshop on Mathematical Theory of Networks and Systems, Leuven, Belgium, July 2004.

The author is with the Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109-2122 USA (e-mail: teneketzis@eecs.umich.edu).

Communicated by M. Effros, Associate Editor for Source Coding.
Digital Object Identifier 10.1109/TIT.2006.880067

pulse code modulation, adaptive differential pulse code modulation) was established in [31], [32].

Properties of real-time decoders for communication systems with noisy channels and Markov sources were discovered in [16], [17].

The model and work most relevant to this paper have appeared in [54], where a zero-delay joint source-channel coding of individual sequences is considered in the presence of a general known noisy channel. The model of [54] considers large time horizons and a performance criterion expressed by the average-per-unit-time additive distortion between the input and output sequences. The authors of [54] describe a coding scheme that asymptotically performs, on all individual sequences, as well as the best among a finite set of schemes.

In this paper, we discover the structure of optimal real-time encoders and decoders for communication systems consisting of Markov sources, noisy channels without any feedback from the output of the channels to the encoder, and general additive distortion measures. The results of this paper are different from those of: [3] where necessary conditions for optimality of real-time encoders and decoders are stated; [15] where attention is restricted to myopic policies; and [30] where attention is restricted to memoryless sources. Our problem formulation and results are also different from those of [4]–[7], [45], [68] as the real-time requirement in our problem differs from the causality requirement in [4]–[6], [45], [68] (cf [4]). In our model the encoder has imperfect knowledge of the information available to the receiver(s)/decoder(s). Thus, the situation is different from that considered in [2], [7]–[15], [50]–[52], [18]–[21], where at each time instant the encoder has perfect knowledge of the receiver's information. Our objectives, hence our results, are different from those of [31], [32], [46], [53]. We are interested in the structure of optimal real-time encoders and decoders, therefore, our approach to and results on real-time communication problems are different from the bounds derived in [22]–[29] and the properties of real-time decoders in [16], [17]. Our structural results on real-time encoding-decoding hold for any finite time horizon as opposed to [54] where the results on real-time encoding-decoding are developed for a large time horizon. Our approach and that of [54] to real-time encoding-decoding are complementary. Our approach is decision-theoretic and provides insight into the structure of real-time encoders and decoders. The approach in [54] is based on coding ideas and provides insight into the construction of real-time coding schemes that work well for large time horizons. Finally, because of the real-time constraint on encoding and decoding, our approach and results on the broadcast system and the Wyner–Ziv problem are distinctly different from those of [33], [37]–[44], and [34], respectively.

The main contribution of this paper is the determination of the structure of optimal real-time encoding and decoding strategies for the following classes of systems. 1) The point-to-point communication system consisting of a Markov source, a noisy channel without feedback, a receiver with limited memory, and a general additive distortion measure. 2) The broadcast system ([33]) with Markov sources and general, additive distortion measures. 3) A real-time variation of the Wyner–Ziv problem ([34]). Our philosophical approach to determining the structure of optimal real-time encoders and decoders is similar to that of [2], [7]. Real-time encoding is conceptually the "difficult" part of the overall problem. For point-to-point communication systems we prove that if the source is kth-order Markov, one may, without loss of optimality, assume that the encoder forms each output based only on the last k source symbols and its knowledge of the probability distribution on the present state of the receiver's memory. For $k = 1$ our results generalize those of [2] and [7] which state that for a first-order Markov source, one may, without loss of optimality, restrict attention to encoders that form each output based only on the last source symbol and the present state of the receiver's/decoder's memory. For $k > 1$ our results generalize those of [7]. We obtain results on the structure of optimal real-time encoders similar to the above for the real-time broadcast system and a real-time variation of the Wyner–Ziv problem. Our results on the structure of optimal real-time decoders with limited memory are similar to those of [2] where decoders with unlimited memory are considered.

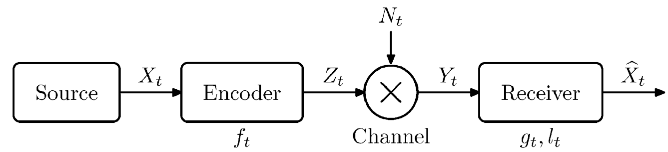

Fig. 1. The point-to-point communication system.

The remainder of the paper is organized as follows. In Sections II, III, IV, and V, we present results on the structure of optimal real-time encoders and decoders for the point-to-point communication system, extensions to continuous state sources and channels and higher order Markov sources, the broadcast system, and a variation of the Wyner–Ziv problem, respectively. We conclude in Section VI.

II. The Real-Time Point-to-Point Communication Problem

Our results on the real-time point-to-point communication system (shown in Fig. 1) are initially developed for the case where the source is first-order finite-state Markov, and the noise in the channel is a discrete-valued random process consisting of mutually independent random variables that are also independent of the source sequence. This simple model allows us to illustrate clearly the key conceptual issues that determine the structure of real-time encoding and decoding strategies. The results developed for the aforementioned model are shown to hold for kth-order finite-state Markov sources, and for continuous state, discrete-time Markov sources, and channels where the noise is described by a sequence of independent continuous-state random variables that are also independent of the source sequence.

A. Problem Formulation

1) The Model: We consider a first-order Markov source X that produces a random sequence $X_1, X_2, \ldots, X_T$. For each $t \in \{1, 2, \ldots, T\}, X_t \in \mathcal{D} := \{1, 2, \ldots, D\}$. The Probability Mass Function (PMF) of X_1, denoted by P_{X_1}, as well as the transition probabilities $P_{X_{t+1}|X_t}(x_{t+1}|x_t), x_t, x_{t+1} \in \mathcal{D}, t = 1, 2, \ldots, T$, are given. For notational simplicity we set $P_{X_{t+1}|X_t}(x_{t+1}|x_t) = P_t(x_{t+1}|x_t)$.

At each time t a signal Z_t taking values in $\mathcal{K} := \{1, 2, \ldots, K\}$, is transmitted to a receiver. The signal Z_t is produced by a real-time encoder, which for every t is characterized by

$$f_t : \mathcal{D}^t \to \mathcal{K}, \tag{1}$$

so that, in general

$$Z_t = f_t(X_1, X_2, \ldots, X_t). \tag{2}$$

The signal Z_t is transmitted to a receiver through a noisy channel. At time t the channel noise is described by a random variable N_t taking values in $\mathcal{Q} := \{1, 2, \ldots, Q\}$. The random variables $N_1, N_2, \ldots, N_T$ are assumed to be mutually independent, and each N_t has a known PMF denoted by $P_{N_t}, t = 1, 2, \ldots, T$. Furthermore, each $N_t, t = 1, 2, \ldots, T$, is independent of $X_1, X_2, \ldots, X_T$.

The signal Y_t, received by the receiver at time t, is a noise-corrupted version of Z_t, that is

$$Y_t = h_t(Z_t, N_t) \tag{3}$$

where h_t is a known function that describes the channel at time t, and for each t Y_t takes values in the set $\mathcal{L} = \{1, 2, 3, \ldots, L\}$.

The receiver has limited memory, which is updated as follows:

1) At $t = 1$ only Y_1 is available, and a discrete random variable

$$W_1 = l_1(Y_1), \tag{4}$$

taking values in $\mathcal{M} = \{1, 2, \ldots, M\}$, is stored in memory.

2) At $t = 2, 3, \ldots, T$, the memory is updated according to the rule

$$W_t = l_t(W_{t-1}, Y_t) \tag{5}$$

where W_t takes values in $\mathcal{M}$, and $l_t, t = 2, 3, \ldots, T$, are given functions.

At $t = 1, 2, \ldots, T$, the receiver generates a variable $\hat{X}_t \in \mathcal{D}$ by the rule

$$\hat{X}_1 = g_1(Y_1) \tag{6}$$

$$\hat{X}_t = g_t(W_{t-1}, Y_t), \quad t = 2, \ldots, T \tag{7}$$

where

$$g_1 : \mathcal{L} \to \mathcal{D} \tag{8}$$

and

$$g_t : \mathcal{M} \times \mathcal{L} \to \mathcal{D}. \tag{9}$$

The random variable $\hat{X}_t$ is an approximation of X_t.

2) The Performance Criterion: For each $t = 1, 2, \ldots, T$, a function

$$\rho_t : \mathcal{D} \times \mathcal{D} \to [0, \infty) \tag{10}$$

is given, and

$$\mathcal{J}_t = E\{\rho_t(X_t, \hat{X}_t)\} \tag{11}$$

measures the average distortion at t. The system's performance is measured by

$$\mathcal{J} = \sum_{t=1}^{T} \mathcal{J}_t = \sum_{t=1}^{T} E\{\rho_t(X_t, \hat{X}_t)\} \tag{12}$$

The expectation in (12) is with respect to a probability measure that is determined by the distribution of the sequence $X_1, X_2, \ldots, X_T$, the choice of the functions f_t, l_t, g_t, the channel h_t, and the statistics of the noise $N_t, t = 1, 2, \ldots, T$.

3) The Optimization Problem (Problem (P)): It is assumed that the model of Section II.A1 and the performance criterion of Section II.A2 are common knowledge ([48], [69]) to the encoder and the receiver/decoder.

Under this assumption the optimization problem (Problem (P)) under consideration is the following:

Problem (P): Consider the model of Section II.A.1. Given $T, \mathcal{D}, \mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{Q}, h_t, \rho_t, l_t, P_{N_t}, t = 1, 2, \ldots, T, P_{X_1}, P_t, t = 1, 2, \ldots, T-1$, choose the functions $f_1, f_2, \ldots, f_T$ and $g_1, g_2, g_3, \ldots, g_T$ to minimize $\mathcal{J}$, given by (12).

Note that in Problem (P) the memory update rule $l := (l_1, l_2, \ldots, l_T)$ is fixed and given. Furthermore, by assumption, it is of the form (4)–(5). The analysis and results that follow are derived under the above assumption on l.

We proceed with the analysis of Problem (P) as follows. We first determine the structure of optimal real-time encoding rules for any fixed arbitrary decoding rule. Then, we determine the structure of optimal real-time decoding rules for any fixed arbitrary encoding rule.

B. The Structure of Optimal Real-Time Encoders

We show that for first-order Markov sources the solution to the real-time encoding optimization problem can be obtained by restricting attention to encoding rules that depend on the source's current state and the PMF (according to the encoder's perception) of the receiver's memory. Before we proceed with the statement of the main result of this section (Theorem 1) we introduce the following concepts and notation.

Definition 1: A *design* is called a choice of a system of functions $f_1, f_2, \ldots, f_T, l_1, l_2, \ldots, l_T, g_1, g_2, \ldots, g_T$.

Let $\mathcal{P}^{\mathcal{M}}$ denote the space of PMFs on the set $\mathcal{M}$, and $P_{W_t} \in \mathcal{P}^{\mathcal{M}}$ denote the PMF of the random variable $W_t, t = 1, 2, \ldots, T$,

$$P_{W_t} := (P_{W_t}(1), P_{W_t}(2), \ldots, P_{W_t}(M)). \tag{13}$$

The PMF P_{W_g} gives the encoder's perception of the decoder's state (i.e., the state of the decoder's memory) at time t.

Given a design d, and any realization $x_1, x_2, \ldots, x_T$ of $X_1, X_2, \ldots, X_T$, the PMF P_{W_t} is well-defined for all $t = 1, 2, \ldots, T$.

Definition 2: Consider a design. The encoder $f := (f_1, f_2, \ldots, f_T)$ is said to be *separated* if for every $t, t = 2, 3, \ldots, T$,

$$f_t : \mathcal{D} \times \mathcal{P}^{\mathcal{M}} \to \mathcal{K}.$$

Notation: For the rest of the paper we adopt the following notation. We denote by $E^d(\cdot)$ the expectation with respect to the probability measure determined by the design d. We denote by P^d (respectively, P^γ) the probability measure determined by the design d (respectively, the component γ of a design d).

The main result of this section is provided by the following theorem.

Theorem 1: In Problem (P) (Section II-A3) there is no loss of optimality if one restricts attention to designs consisting of separated encoding policies.

We present two approaches to proving Theorem 1. The first approach follows the philosophy of [7]. The second approach is based on Markov decision theory and follows the philosophy of [2].

We begin with the first approach. We first establish the noisy-transmission analogues of two fundamental lemmata of [7], namely, the two-stage lemma and the three-stage lemma. Using the results of these lemmata we prove the assertion of Theorem 1 by induction and the method of "repackaging" of random variables.

1) The Two-Stage Lemma: Consider the problem formulated in Section II-A with $T = 2$, and any joint distribution of the random vector (X_1, X_2). At the beginning of stage 2 the content of the receiver's memory is

$$\begin{aligned} W_1 &= l_1(Y_1) = l_1(h_1(Z_1, N_1)) \\ &= l_1(h_1(f_1(X_1), N_1)) = \hat{l}_1(X_1, N_1) \end{aligned} \tag{14}$$

Furthermore

$$Z_2 = f_2(X_1, X_2), \tag{15}$$

$$\begin{aligned} Y_2 &= h_2(Z_2, N_2) = h_2(f_2(X_1, X_2), N_2) \\ &= \hat{h}_2(X_1, X_2, N_2) \end{aligned} \tag{16}$$

and

$$W_2 = l_2(W_1, Y_2) = l_2(W_1, h_2(Z_2, N_2)). \tag{17}$$

Lemma 1: Consider a two-stage system with a design d where

$$f_2 : \mathcal{D}^2 \to \mathcal{K} \tag{18}$$

so that

$$Z_2 = f_2(X_1, X_2). \tag{19}$$

Then one can replace f_2 with $\hat{f}_2$

$$\hat{f}_2 : \mathcal{D} \times \mathcal{P}^{\mathcal{M}} \to \mathcal{K} \tag{20}$$

so that

$$Z_2 = \hat{f}_2(X_2, P_{W_1}) \tag{21}$$

and the resulting new design $\hat{d}$ is at least as good as the old design.

Proof: See Appendix I. □

2) The Three-Stage Lemma: Consider the problem formulated in Section II-A with $T = 3$ and any joint distribution of the random vector (X_1, X_2, X_3). For any design $d := (f_1, f_2, f_3, l_1, l_2, l_3, g_1, g_2, g_3)$ define the resulting cost

$$\mathcal{J}^d := \mathcal{J}_1^d + \mathcal{J}_2^d + \mathcal{J}_3^d, \tag{22}$$

where $\mathcal{J}_i^d, i = 1, 2, 3$, is given by (11). Consider a design $d' := (f_1, f_2, f_3, l_1, l_2, l_3, g_1, g_2, g_3)$ where f_3 is a separated encoder (cf. Definition 2), whereas f_2 is not. The following result holds.

Lemma 2: Consider a three-stage system with the design d'. One can replace d' with another design $\hat{d}' := (f_1, \hat{f}_2, f_3, l_1, l_2, l_3, g_1, g_2, g_3)$ where $\hat{f}_2$ is a separated encoder, and the new design is at least as good as the old design, that is

$$\mathcal{J}^{\hat{d}'} \leq \mathcal{J}^{d'}. \tag{23}$$

Proof: See Appendix II. □

3) Proof of the Main Result: We complete the proof of the main result (Theorem 1) based on the two-stage lemma and the three-stage lemma. We proceed by induction. The following lemma establishes the basis of the induction process.

Lemma 3: Consider the problem formulated in Section II-A. Then for any design

$$d := (f_1, f_2, \ldots, f_T, l_1, l_2, \ldots, l_T, g_1, g_2, \ldots, g_T) \tag{24}$$

where $f_t, t = 1, 2, \ldots T$, is of the general form (2), one can replace the last encoder f_T by one of the form

$$\hat{f}_T : \mathcal{D} \times \mathcal{P}^{\mathcal{M}} \to \mathcal{K} \tag{25}$$

without any performance loss.

Proof: See Appendix III. □

Lemma 3 establishes the basis of the induction process. To prove the induction step, consider a design $d' = (f'_1, \ldots, f'_T, l'_1, \ldots, l'_T, g'_1, \ldots, g'_T)$, and suppose that $f'_{t+1}, f'_{t+2}, \ldots, f'_T$ are separated encoders (cf. Definition 2). We must show that f'_t can be replaced by an encoder $\hat{f}'_t$ that is separated and is such that the performance of the new design $\hat{d}' := (f'_1, \ldots, f'_{t-1}, \hat{f}'_t, f'_{t+1}, \ldots, f'_T, l'_1, \ldots, l'_T, g'_1, \ldots, g'_T)$ is at least as good as that of d'. For that matter, the T-stage system can be viewed as a three-stage system where the encoder at the third stage is separated and the source is first-order Markov. This can be done as follows. Define

$$\bar{X}_1 := (X_1, X_2, \ldots, X_{t-1}), \tag{26}$$

$$\bar{X}_2 := X_t, \tag{27}$$

$$\bar{X}_3 := (X_{t+1}, X_{t+2}, \ldots, X_T), \tag{28}$$

$$\bar{N}_1 := (N_1, N_2, \ldots, N_{t-1}), \tag{29}$$

$$\bar{N}_2 := N_t, \tag{30}$$

$$\bar{N}_3 := (N_{t+1}, N_{t+2}, \ldots, N_T), \tag{31}$$

$$\bar{Y}_1 := (Y_1, Y_2, \ldots, Y_{t-1}), \tag{32}$$

$$\bar{Y}_2 := Y_t, \tag{33}$$

$$\bar{Y}_3 := (Y_{t+1}, Y_{t+2}, \ldots, Y_T), \tag{34}$$

$$\bar{W}_1 := W_{t-1} = \bar{\phi}(\bar{X}_1, \bar{N}_1), \tag{35}$$

where $\bar{\phi}$ is specified in terms of $f'_1, f'_2, \ldots, f'_{t-1}, l'_1, l'_2, \ldots, l'_{t-1}, h_1, h_2, \ldots, h_{t-1}$,

$$\bar{W}_2 := \bar{l}'_2(\bar{W}_1, \bar{Y}_2) = l'_t(W_{t-1}, Y_t) = W_t, \tag{36}$$

$$\hat{\bar{X}}_1 := (\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_{t-1}), \tag{37}$$

$$\hat{\bar{X}}_2 := \hat{X}_t, \quad (38)$$
$$\hat{\bar{X}}_2 := \bar{g}_2(\bar{W}_1, \bar{Y}_2) = g_t(W_{t-1}, Y_t), \quad (39)$$
$$\hat{\bar{X}}_3 := (\hat{X}_{t+1}, \hat{X}_{t+2}, \ldots, \hat{X}_T), \quad (40)$$
$$\bar{Z}_1 := (Z_1, Z_2, \ldots, Z_{t-1}), \quad (41)$$
$$\bar{Z}_2 := Z_t, \quad (42)$$
$$\bar{Z}_3 := (Z_{t+1}, Z_{t+2}, \ldots, Z_T), \quad (43)$$
$$\bar{\rho}_1(\bar{X}_1, \hat{\bar{X}}_1) := \sum_{s=1}^{t-1} \rho_s(X_s, \hat{X}_s), \quad (44)$$
$$\bar{\rho}_2(\bar{X}_2, \hat{\bar{X}}_2) := \rho_t(X_t, \hat{X}_t), \quad (45)$$
$$\bar{\rho}_3(\bar{X}_3, \hat{\bar{X}}_3) := \sum_{s=t+1}^{T} \rho_s(X_s, \hat{X}_s). \quad (46)$$

The encoder f'_t at time t has the structure

$$Z_t = f'_t(X_1, X_2, \ldots, X_t) \quad (47)$$

which translates to

$$\bar{Z}_2 = \bar{f}_2(\bar{X}_1, \bar{X}_2) \quad (48)$$

The source $(\bar{X}_1, \bar{X}_2, \bar{X}_3)$ is first-order Markov, because $\bar{X}_1 = (X_1, X_2, \ldots, X_{t-1})$ and $\bar{X}_3 = (X_{t+1}, X_{t+2}, \ldots, X_T)$ are conditionally independent given $\bar{X}_2 = X_t$, as the original source is first-order Markov.

For the three-stage system defined above we claim the following.

Claim: Since the encoders at stages $t+1, t+2, \ldots T-1, T$ are separated, they define

$$\bar{Z}_3 = \tilde{f}_3(\bar{X}_3, P_{\bar{W}_2}) \quad (49)$$

for some function $\tilde{f}_3$.

Assuming for the moment that the above claim is true, the three-stage system defined above satisfies the conditions of the three-stage lemma. Consequently, by Lemma 2, the encoder $\bar{f}_2$, can be replaced by one that has the form

$$\bar{Z}_2 = \hat{\bar{f}}_2(\bar{X}_2, P_{\bar{W}_1}) \quad (50)$$

and the resulting new design performs at least as well as the one it replaces. In the original notation, (50) corresponds to an encoder $\hat{f}'_t$ that has the structure

$$Z_t = \hat{f}'_t(X_t, P_{W_{t-1}}),$$

and is such that the design $\hat{d}'$ is at least as good as d'.

To complete the proof of the induction step we must verify that the claim expressed by (49) is true.

Proof of Claim (49): To prove (49) we note that

$$\bar{Z}_3 := (Z_{t+1}, Z_{t+2}, \ldots, Z_{T-1}, Z_T) \quad (51)$$

Furthermore, by assumption

$$Z_s = f'_s(X_s, P_{W_{s-1}}) \quad (52)$$

for all $s > t$. In addition, for any $b \in \mathcal{M}$, any $X_{s+1} = x, x \in \mathcal{D}$ and any given $\tilde{P}_{W_s} \in \mathcal{P}^{\mathcal{M}}$ we have

$$\begin{aligned}
P\left(W_{s+1} = b \,\middle|\, X_{s+1} = x, \tilde{P}_{W_s}\right)
&= P\left(l'_{s+1}(Y_{s+1}, W_s) = b \,\middle|\, X_{s+1} = x, \tilde{P}_{W_s}\right)\\
&= P\left(l'_{s+1}(h_{s+1}(Z_{s+1}, N_{s+1}), W_s)\right.\\
&\quad \left.= b \,\middle|\, X_{s+1} = x, \tilde{P}_{W_s}\right)\\
&= P\left(l'_{s+1}\left(h_{s+1}\left(f'_{s+1}\left(X_{s+1}, \tilde{P}_{W_s}\right), N_{s+1}\right), W_s\right)\right.\\
&\quad \left.= b \,\middle|\, X_{s+1} = x, \tilde{P}_{W_s}\right)\\
&= \sum_{(n,w') \in A\left(x, \tilde{P}_{W_s}\right)} P\left(N_{s+1} = n, W_s = w' \,\middle|\, \tilde{P}_{W_s}\right)\\
&= \sum_{(n,w') \in A\left(x, \tilde{P}_{W_s}\right)} P(N_{s+1} = n) P(W_s = w')\\
&= P\left(A\left(x, \tilde{P}_{W_s}\right)\right)
\end{aligned} \quad (53)$$

where

$$A\left(x, \tilde{P}_{W_s}\right) = \Big\{(n, w') \in \mathcal{Q} \times \mathcal{M}: l_{s+1}\left(h_{s+1}\left(f_{s+1}\left(x, \tilde{P}_{W_s}\right), n\right), w'\right) = b\Big\}. \quad (54)$$

The fifth equality in (53) holds, because the random variables N_{s+1}, W_s are independent.

From (53) we conclude that

$$P_{W_{s+1}} = \Gamma_{s+1}(X_{s+1}, P_{W_s}) \quad (55)$$

for some function Γ_{s+1}.

Hence, (52) and (55) combined give, for $s > t$

$$Z_s = f^*_s(X_s, X_{s-1}, \ldots, X_{t+1}, P_{W_t}) \quad (56)$$

for some function f^*_s.

Combining (28), (36), (51) and (56) we obtain

$$\begin{aligned}
\bar{Z}_3 &:= (Z_{t+1}, Z_{t+2}, \ldots, Z_{T-1}, Z_T)\\
&= \tilde{f}_3(X_{t+1}, X_{t+2}, \ldots, X_{T-1}, X_T, P_{W_t})\\
&= \tilde{f}_3(\bar{X}_3, P_{\bar{W}_2})
\end{aligned} \quad (57)$$

for some function $\tilde{f}_3$.

This completes the proof of claim (49), the proof of the induction step, and the proof of Theorem 1. □

C. Discussion of the Main Result on Real-Time Encoding

Theorem 1 provides a qualitative result on the structure of optimal real-time "noisy" encoders for Markov sources. If M, the number of discrete values $W_t (t = 1, 2, \ldots, T)$ can take, is small compared to T, then the result of Theorem 1 provides a substantial simplification of the optimal encoder design problem for the following reasons. For large T the (on-line) implementation of real-time encoders of the form

$$Z_t = f_t(X_1, X_2, \ldots, X_t) \quad (58)$$

requires a large memory. Moreover, the memory requirements on the encoder's site change as the finite horizon T, over which Problem (P) is being considered, changes. The result of Theorem 1 implies that the use of separated encoding strategies does not entail any loss of optimality for Problem (P), it requires a finite memory of size M on the encoder's site, and this memory size is independent of T. Furthermore, it will become evident from the following discussion (cf. (59)–(60)) that, as a consequence of Theorem 1, the determination of optimal real-time encoding strategies can be achieved using the computational methods available for the solution of Partially Observed Markov Decision Processes (POMDPs). There is a significant amount of literature devoted to the computation of optimal strategies for POMDPs and to approximating the value function of POMDPs (see [35], [47], [49], [55]–[67], and references therein).

The result of Theorem 1 can be intuitively explained as follows. When the receiver's memory update functions $l_1, l_2, \ldots, l_T$ and decision functions $g_1, g_2, \ldots, g_T$ are fixed, the real-time encoding problem can be viewed as a centralized stochastic control problem where the encoder controls the PMF of the receiver's memory. For this reason ([36]) an optimal real-time encoding rule can be determined by backward induction. The optimality equations are, for any $x \in \mathcal{D}, \tilde{P}_W \in \mathcal{P}^{\mathcal{M}}$, (see [36, Ch. 6])

$$V_{T+1}\left(x, \tilde{P}_{W_T}\right) = 0$$

$$V_t\left(x, \tilde{P}_{W_{t-1}}\right) = \min_{z \in \mathcal{K}} \left\{ E_z(\rho_t(x, \hat{X}_t)) + \sum_{x_{t+1} \in \mathcal{D}} P_{X_{t+1} \mid X_t}(x_{t+1}|x) V_{t+1}\left(x_{t+1}, \bar{P}^z_{W_t}\right) \right\} \tag{59}$$

$$t = 1, 2, \ldots, T \tag{60}$$

where $\tilde{P}_{W_0} := \emptyset$ and this indicates that the receiver's memory is empty

$$\begin{aligned} E_z(\rho_t(x, \hat{X}_t)) &= \sum_{b \in \mathcal{D}} \rho_t(x, b) P\left(\hat{X}_t = b \,\middle|\, x, \tilde{P}_{W_{t-1}}, z\right) \\ &= \sum_{b \in \mathcal{D}} \rho_t(x, b) P\Big((w, n) \in \mathcal{M} \times \mathcal{Q}: \\ &\qquad g_t(h_t(z, n), w) = b \,\Big|\, \tilde{P}_{W_{t-1}}, z\Big) \end{aligned} \tag{61}$$

and the components of $\bar{P}^z_{W_t}$ are given by

$$\begin{aligned} \bar{P}^z(W_t = c) = P\Big((w, n) \in \mathcal{M} \times \mathcal{Q}:& \\ l_t(h_t(z, n), w) = c \,\Big|\, z, \tilde{P}_{W_{t-1}}\Big),& \quad c \in \mathcal{M}. \end{aligned} \tag{62}$$

A further formal explanation of the optimality equations (59)–(60) will be provided in Section II.D, where an alternative proof of Theorem 1 will be presented.

We now compare the key features of our problem with those of the problems investigated in [7], [8], [2], and [18]. This comparison, together with the discussion of the preceding paragraph, provides additional insight into the nature of Problem (P). In [7], [8] communication is noiseless, therefore, for fixed $l_1, l_2, \ldots, l_T$, once $f_1, f_2, \ldots f_T$ are specified the encoder knows at every instant of time the state of the receiver's memory. Thus, when $l_1, l_2, \ldots, l_T$ and $g_1, g_2, \ldots, g_T$ are fixed, the encoder's task is to choose $f_1, f_2, \ldots, f_T$ so as to control the receiver's memory and to minimize a cost function of the form (12). In [2], [18] the channel is noisy, but there is a noiseless feedback from the output of the channel to the encoder so that the encoder knows at every instant of time the state of the receiver's memory. Hence, for fixed $l_1, l_2, \ldots, l_T$ and $g_1, g_2, \ldots, g_T$ the encoder's problem in [2], [18] is essentially the same as its problem in [7], [8]. In our problem the encoder does not know the state of the receiver's memory. However, for fixed memory update functions $l_1, l_2, \ldots, l_T$ and fixed decision functions $g_1, g_2, \ldots, g_T$, given the encoder's decisions $z_1, z_2, \ldots, z_T$, the encoder knows the probability distribution of the receiver's memory at any t, and the probability distribution of the receiver's decisions at any t. Thus, the encoder's task is to control, through the choice of $Z_1, Z_2, \ldots, Z_T$, the distribution of the receiver's memory so as to minimize a performance criterion given by (12).

The observation that the real-time encoding problem can be viewed as a centralized stochastic control problem where the encoder controls the PMF of the receiver's memory leads to another approach to the problem, which we discuss next.

D. An Alternative Proof of the Main Result on Real-Time Encoding

Consider any (fixed) memory update rule $l := (l_1, l_2, \ldots, l_T)$ and any arbitrary (but fixed) decision rule $g := (g_1, g_2, \ldots, g_T)$ for the decoder. Define the process $\{R_t, t = 1, 2, \ldots, T\}$ by

$$R_1 = X_1 \tag{63}$$

$$R_t = (X_t, P_{W_{t-1}}), t = 2, \ldots, T \tag{64}$$

where $P_{W_t}, t = 1, 2, \ldots, T$, is defined by (13).

Lemma 4: The process $\{R_t, t = 1, 2, \ldots, T\}$ is conditionally Markov given the Z_t's; that is, for any r^t, z^t

$$P(R_{t+1} \mid r^t, z^t) = P(R_{t+1} \mid r_t, z_t) \tag{65}$$

where

$$r^t := (r_1, r_2, \ldots, r_t), \tag{66}$$

$$z^t := (z_1, z_2, \ldots, z_t). \tag{67}$$

Proof: For any realization $x^t, z^t, \tilde{P}_{W_1}, \ldots, \tilde{P}_{W_{t-1}}$ of $X^t, Z^t, P_{W_1}, P_{W_2}, \ldots, P_{W_{t-1}}$, respectively, where

$$X^t := (X_1, X_2, \ldots, X_t), \tag{68}$$

we have

$$P\left(X_{t+1} = y \,\middle|\, x^t, z^t, \tilde{P}_{W_1}, \tilde{P}_{W_2}, \ldots, \tilde{P}_{W_{t-1}}\right) = P(X_{t+1} = y \mid x_t) \tag{69}$$

for any $y \in \mathcal{D}$ by the first-order Markov property of the source. Furthermore, for any $t > 1$ and $b \in \mathcal{M}$

$$
\begin{aligned}
&P\left(W_t = b \,\middle|\, x^t, z^t, \tilde{P}_{W_1}, \tilde{P}_{W_2}, \ldots, \tilde{P}_{W_{t-1}}\right)\\
&\quad = P\left(l_t(Y_t, W_{t-1}) = b \,\middle|\, x^t, z^t, \tilde{P}_{W_1}, \tilde{P}_{W_2}, \ldots, \tilde{P}_{W_{t-1}}\right)\\
&\quad = P\left(l_t(h_t(Z_t, N_t), W_{t-1}) = b \,\middle|\, x^t, z^t, \tilde{P}_{W_1}, \ldots, \tilde{P}_{W_{t-1}}\right)\\
&\quad = P\left((n, w') \in \mathcal{Q} \times \mathcal{M} : l_t(h_t(z_t, n), w') = b \,\middle|\, z_t, \tilde{P}_{W_{t-1}}\right)\\
&\quad = P\left(W_t = b \,\middle|\, z_t, \tilde{P}_{W_{t-1}}\right); \qquad (70)
\end{aligned}
$$

for $t = 1$ and $b \in \mathcal{M}$

$$
\begin{aligned}
&P(W_1 = b_1 \,|\, z_1, x_1) = P(l_1(h_1(Z_1, N_1)) = b \,|\, z_1, x_1)\\
&\quad = P(l_1(h_1(Z_1, N_1)) = b | z_1)\\
&\quad = P(W_1 = b | z_1). \qquad (71)
\end{aligned}
$$

The third equality in (70) and the second equality in (71) hold because, by assumption, $N_1, N_2, \ldots, N_T$ is a sequence of independent random variables that are also independent of $X_1, X_2, \ldots, X_T$; therefore, N_t is independent of $X^t, Z^t, W_1, W_2, \ldots, W_{t-1}$ and N_1 is independent of X_1, Z_1.

From (70) and (71) we conclude that, for $t > 1$

$$P_{W_t} = \mu_t\left(z_t, \tilde{P}_{W_{t-1}}\right) \qquad (72)$$

and for $t = 1$

$$P_{W_1} = \mu_1(z_1) \qquad (73)$$

where $\mu_1, \mu_t, t = 2, 3, \ldots, T$, are functions determined by (71) and (70), respectively. Therefore, because of (69) and (72) we obtain for $t > 1$ and any $y \in \mathcal{D}, P'_{W_t} \in \mathcal{P}^{\mathcal{M}}$,

$$
\begin{aligned}
&P\left(R_{t+1} = \left(x', P'_{W_t}\right) \,|\, r^t, z^t\right)\\
&\quad = P(X_{t+1} = x' \,|\, x_t)\\
&\qquad \times \delta\left(P'_{W_t}, \mu_t\left(\tilde{P}_{W_{t-1}}, z_t\right)\right)\\
&\quad = P\left(R_{t+1} = \left(x', P'_{W_t}\right) \,\middle|\, x_t, \tilde{P}_{W_{t-1}}, z_t\right)\\
&\quad = P\left(R_{t+1} = \left(x', P'_{W_t}\right) \,|\, r_t, z_t\right) \qquad (74)
\end{aligned}
$$

where $\delta(\cdot)$ is the Kronecker delta, i.e.

$$\delta(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases} \qquad (75)$$

Equation (74) proves the assertion of Lemma 4. □

The conditional Markov property of $\{R_t, t = 1, 2, \ldots, T\}$ implies that for each t and each realization $x^t, z^t, \tilde{P}_{W_1}, \tilde{P}_{W_2}, \ldots, \tilde{P}_{W_{t-1}}$, of $X^t, Z^t, P_{W_1}, P_{W_2}, \ldots, P_{W_{t-1}}$

$$
\begin{aligned}
\mathcal{J}_t &= E\left\{\rho_t(X_t, \hat{X}_t) \,\middle|\, x^t, z^t, \tilde{P}_{W_1}, \tilde{P}_{W_2}, \ldots, \tilde{P}_{W_{t-1}}\right\}\\
&= \sum_b P\left(\hat{X}_t = b \,\middle|\, x^t, z^t, \tilde{P}_{W_1}, \tilde{P}_{W_2}, \ldots, \tilde{P}_{W_{t-1}}\right) \rho_t(x_t, b)\\
&= \sum_b P\left((n, w') \in \mathcal{Q} \times \mathcal{M} : g_t(h_t(z_t, n), w') = b \,\middle|\, x^t, z^t, \tilde{P}_{W_1}, \tilde{P}_{W_2}, \ldots, \tilde{P}_{W_{t-1}}\right) \rho_t(x_t, b)\\
&= \sum_b P\left((n, w') \in \mathcal{Q} \times \mathcal{M} : g_t(h_t(z_t, n), w') = b \,\middle|\, z_t, \tilde{P}_{W_{t-1}}\right) \rho_t(x_t, b)\\
&= \tilde{\rho}_t(x_t, \tilde{P}_{W_{t-1}}, z_t) \qquad (76)
\end{aligned}
$$

for some function $\tilde{\rho}_t$. Because of (76) we obtain

$$
\begin{aligned}
\mathcal{J} &= E\left\{\sum_{t=1}^{T} \rho_t(X_t, \hat{X}_t)\right\}\\
&= E\left\{\sum_{t=1}^{T} \left\{E\left\{\rho_t(X_t, \hat{X}_t) \,|\, X^t, Z^t, P_{W_1}, \ldots, P_{W_{t-1}}\right\}\right\}\right\}\\
&= E\left\{\sum_{t=1}^{T} \tilde{\rho}_t\left(X_t, Z_t, P_{W_{t-1}}\right)\right\}\\
&= E\left\{\sum_{t=1}^{T} \tilde{\rho}_t(R_t, Z_t)\right\}. \qquad (77)
\end{aligned}
$$

Consequently, for fixed $l := l_1, l_2, \ldots, l_T$ and $g := g_1, g_2, \ldots, g_T$, the problem is to control through the choice of Z_t, for all t, the transition probabilities from R_t to R_{t+1} so as to minimize the cost given by (77). From Markov decision theory (e.g., [36], Chapter 6) it is well known that an optimal control law, i.e., an optimal encoding rule, is of the form

$$Z_t = \hat{f}_t(R_t) = \hat{f}_t\left(X_t, P_{W_{t-1}}\right) \qquad (78)$$

for all t, and that an optimal encoding rule can be determined by the solution of the dynamic program described by (59)–(62).

E. The Structure of Optimal Real-Time Decoders

Let $\Xi(\mathcal{D})$ denote the set of PMFs on $\mathcal{D}$. Consider arbitrary (but fixed) encoding and memory updating strategies $f := (f_1, f_2, \ldots, f_T)$ and $l := (l_1, l_2, \ldots, l_T)$, respectively, where f_t and l_t are of the general form (1) and (4), (5), respectively. Let P_{X_t} denote the PMF of $X_t, t = 1, 2, \ldots, T$. Let $\xi_t^{f,l}(y, w)$ denote the conditional PMF of X_t given the decoder's information y, w at time t; that is

$$\xi_t^{f,l}(y, w)(x) = P^{f,l}(X_t = x \,|\, Y_t = y, W_{t-1} = w). \qquad (79)$$

The superscripts on both sides of (79) indicate that this conditional PMF explicitly depends on f and l. To proceed further, we need the following.

Definition: For any $\xi \in \Xi(\mathcal{D})$ and $t \geq 1$ define

$$\tau_t(\xi) = \underset{\alpha \in \mathcal{D}}{\arg\min} \sum_{x \in \mathcal{D}} \rho_t(x, a)\xi(x). \qquad (80)$$

With the above notation and definition we present the result that describes the structure of optimal real-time decoders.

Theorem 2: Let f, l, be any (fixed) encoding and memory updating strategies, respectively. The optimal real-time decoding rule for f, l is given by

$$g_1^*(Y_1) = \tau_1\left(\xi_1^{f,l}(Y_1)\right) \tag{81}$$

$$g_t^*(Y_t, W_{t-1}) = \tau_t\left(\xi_t^{f,l}(Y_t, W_{t-1})\right), t = 2, \dots, T. \tag{82}$$

Proof: We make the following observation. For any fixed f, l, minimizing $\mathcal{J}$ (given by (12)) is equivalent to minimizing $\mathcal{J}_t$ (given by (11)) for each t. The assertion of Theorem 2 follows from the above observation and the definition of τ_t (cf. (80)). □

The conditional PMF's $\xi_1^{f,l}(y_1)$ and $\xi_t^{f,l}(y_t, w_{t-1}), t = 2, 3, \dots, T$, can be computed using Bayes' rule, the functional form of f and l, the dynamics of the Markov source, the statistics of the channel noise, and the fact that $N_1, N_2, \dots, N_T$ are mutually independent, and independent of $X_1, X_2, \dots, X_T$. Their computation is presented in Appendix IV.

III. Extensions

As pointed out in Section II, the results of Sections II-B–II-E were developed for a simple model so as to clearly illustrate the key conceptual issues that determine the structure of optimal real-time encoding and decoding strategies. In this section we discuss extensions of these results to more general models.

A. Continuous-State First-Order Markov Sources, Continuous-State Channel Noise

The results of Sections II-B–II-E hold for the following systems. The source is described by a continuous-state first-order Markov source $\{X_t, t = 1, 2, \dots, T\}, X_t \in \Re^n$ for all t, with given statistical description. The noise in the channel is described by a random process $\{N_t, t = 1, 2, \dots, T\}, N_t \in \Re^m$, for all t, where the random variables $N_1, N_2, \dots, N_T$ are mutually independent, each N_t has a known cumulative distribution function, and each N_t is independent of $X_1, X_2, \dots, X_T$. The real-time encoder's output $Z_t, t = 1, 2, \dots, T$, takes values in the set $\mathcal{K}$ defined in Section II-A1, the channel output $Y_t \in \Re^p, t = 1, 2, \dots, T$, and the decoder has limited memory as in the model of Section II-A1. The decoder's decisions $\hat{X}_t \in \Re^n, t = 1, 2, \dots, T$, and for each t the distortion measure ρ_t is defined as

$$\rho_t : \Re^n \times \Re^n \to [0, \infty). \tag{83}$$

For any design d (cf. Definition 1) the system's performance is measured by a criterion of the form (12).

For the above model, the results of Theorems 1 and 2 can be proved by the same technical approach as in Sections II-B–II-D and II-E, respectively.

B. Kth-Order Markov Sources

Consider the model of Section II-A1 with only one modification. The source X is a discrete-time, discrete-state, kth-order Markov source $(k > 1)$; that is, for $t > k$

$$\begin{aligned} &P(X_{t+1} = x \mid X_t = x_t, X_{t-1} = x_{t-1}, \dots \\ &\quad X_{t-k+1} = x_{t-k+1}, \dots, X_0 = x_0) \\ &= P(X_{t+1} = x \mid X_t = x_t, X_{t-1} = x_{t-1}, \dots \\ &\quad X_{t-k+1} = x_{t-k+1}) \end{aligned} \tag{84}$$

for any $x_0, x_1, \dots, x_t, x \in \mathcal{D}$ (where the set $\mathcal{D}$ is defined in Section II-A1). We briefly describe the structure of optimal real-time encoders and decoders for this situation.

The structure of optimal real-time decoders is the same as that of the model of Section II.E, and is described by Theorem 2. The computation of the conditional PMF's $\xi_1^{f,l}(y_1), \xi_t^{f,l}(y_t, w_{t-1}), t = 2, 3, \dots, T$, defined in Section II.E and appearing in the statement of Theorem 2, can be performed in the same way as in Appendix IV using (84).

A result similar to that of Theorem 1 is also true. To state this result precisely, we first need the following definition.

Definition 4: The encoder $f := (f_1, f_2, \dots, f_T), T > k$, is said to be *$k$-separated* if

$$f_t : \mathcal{D}^t \to \mathcal{K}, \quad t = 1, 2, \dots, k \tag{85}$$

and

$$f_t : \mathcal{D}^k \times \mathcal{P}^{\mathcal{M}} \to \mathcal{K}, \quad t = k+1, \dots, T. \tag{86}$$

Consider Problem (P) (cf. Section II-A3) for the model of this section. The following result holds.

Theorem 3: In Problem (P) there is no loss of optimality if one restricts attention to designs consisting of k-separated encoding policies.

Proof: For $T \le k$ the assertion of Theorem 3 is trivial. For $T > k$ and any $t, T > t > k$ the assertion of Theorem 3 can be established as follows.

Define the process

$$\tilde{X}_t = (X_t, X_{t+1}, \dots, X_{t+k-1}), t = 1, 2, \dots, T-k+1. \tag{87}$$

Define

$$\tilde{Z}_1 = (Z_1, Z_2, \dots, Z_k), \tag{88}$$

$$\tilde{Z}_t = Z_{t+k-1}, \quad t = 2, 3, \dots, T-k+1, \tag{89}$$

$$\tilde{N}_1 = (N_1, N_2, \dots, N_k), \tag{90}$$

$$\tilde{N}_t = N_{t+k-1}, \quad t = 2, 3, \dots, T-k+1, \tag{91}$$

$$\tilde{Y}_1 = (Y_1, Y_2, \dots, Y_k), \tag{92}$$

$$\tilde{Y}_t = Y_{t+k-1}, \quad t = 2, 3, \dots, T-k+1, \tag{93}$$

$$\tilde{W}_t = W_{t+k-1}, \quad t = 1, 2, \dots, T-k+1, \tag{94}$$

$$\tilde{\hat{X}}_1 = (\hat{X}_1, \hat{X}_2, \dots, \hat{X}_k), \tag{95}$$

$$\tilde{\hat{X}}_t = \hat{X}_{t+k-1}, \quad t = 2, 3, \dots, T-k+1. \tag{96}$$

The functions relating the above variables are as follows. The encoder is characterized by

$$\tilde{Z}_1 = \tilde{f}_1(\tilde{X}_1) \tag{97}$$

$$\begin{aligned} \tilde{Z}_t &= \tilde{f}_t(\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_t) \\ &= f_{t+k-1}(X_1, \dots, X_{t+k-1}) \\ &\quad t = 2, 3, \dots, T-k+1. \end{aligned} \tag{98}$$

The function $\tilde{f}_1$ summarizes the effect of the first k encoders $f_1, f_2, \dots, f_k$; the functions $\tilde{f}_t$ can be uniquely defined by the arguments presented in [7] (Section V). The receiver's memory update functions are

$$\tilde{W}_1 = \tilde{l}_1(\tilde{Y}_1) \tag{99}$$

$$\tilde{W}_t = \tilde{l}_t(\tilde{W}_{t-1}, \tilde{Y}_t), \quad t = 2, 3, \ldots, T-k+1. \tag{100}$$

The function $\tilde{l}_1$ summarizes the recursive build-up of W_k from $Y_1, Y_2, \ldots, Y_k$ through the use of $l_1, l_2, \ldots, l_k$. The receiver's decisions are described by

$$\hat{\tilde{X}}_1 = \tilde{g}_1(\tilde{Y}_1) \tag{101}$$

$$\hat{\tilde{X}}_t = \tilde{g}_t(\tilde{Y}_t, \tilde{W}_{t-1}) \quad t = 2, 3, \ldots, T-k+1. \tag{102}$$

The function $\tilde{g}_1$ summarizes the actions $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_k$ of the first k decoders through $g_1, g_2, \ldots g_k$ and $l_1, l_2, \ldots, l_k$. The distortion functions $\tilde{\rho}_t, t = 1, 2, \ldots, T-k+1$, are described by

$$\tilde{\rho}_1(\tilde{X}_1, \hat{\tilde{X}}_1) = \sum_{s=1}^{k} \rho_s(X_s, \hat{X}_s) \tag{103}$$

and

$$\tilde{\rho}_t(\tilde{X}_t, \hat{\tilde{X}}_t) = \rho_{t+k-1}(X_{t+k-1}, \hat{X}_{t+k-1}) \quad t = 2, 3, \ldots, T-k+1. \tag{104}$$

With the above definitions we have a first-order Markov process $\{\tilde{X}_t, t = 1, 2, \ldots, T-k+1\}$, a model that is the same as that of Section II-A1, and an optimization Problem (P) similar to that of Sections II-A2 and Sections II-A3. For this system Theorem 1 applies to show that in Problem (P) there is no loss of optimality if one restricts attention to separated encoders $\hat{\tilde{f}} := (\hat{\tilde{f}}_1, \hat{\tilde{f}}_2, \ldots, \hat{\tilde{f}}_{T-k+1})$, that is, encoders of the form

$$\tilde{Z}_1 = \hat{\tilde{f}}_1(\tilde{X}_1) \tag{105}$$

$$\tilde{Z}_t = \hat{\tilde{f}}_t\left(\tilde{X}_t, P_{\tilde{W}_{t-1}}\right) \quad t = 2, 3, \ldots, T-k+1. \tag{106}$$

Reverting to the original notation, (106) corresponds to

$$Z_{t+k-1} = \hat{f}_{t+k-1}\left(X_t, X_{t+1}, \ldots, X_{t+k-1}, P_{W_{t+k-2}}\right) \tag{107}$$

for some function $\hat{f}_{t+k-1}, t = 1, 2, \ldots, T-k+1$, or equivalently

$$Z_t = \hat{f}_t\left(X_t, X_{t-1}, \ldots, X_{t-k+1}, P_{W_{t-1}}\right) \tag{108}$$

for $t = k+1, k+2, \ldots, T$, and this establishes the assertion of Theorem 3. □

IV. The Real-Time Broadcast Problem

For the broadcast system, we show that the structure of optimal real-time encoders and decoders is similar to the one discovered in Section II for the point-to-point communication system.

A. The Model

Consider the system of Fig. 2. Each source $i, i = 1, 2, \ldots, n$, is described by a finite-state discrete-time Markov Chain $\{X_t^i, t = 1, 2, \ldots, T\}$ where $X_t^i \in \mathcal{D}^i := \{1, 2, \ldots D^i\}$ for all t. The initial PMF on X_1^i and the transition functions $P^i_{X^i_{t+1} \mid X^i_t}, t = 1, 2, \ldots, T-1$ are given for all i and for all t. The Markov Chains $\{X_t^i, t = 1, 2, \ldots, T\}$ are assumed to be mutually independent. The message of source i must be communicated in real-time to receiver i.

The output of all sources is encoded by a single encoder. At time t a signal Z_t, taking values in $\mathcal{K} := \{1, 2, \ldots K\}$, is transmitted to all receivers. The signal produced by the real-time encoder is characterized by

$$f_t : \prod_{i=1}^{n} (\mathcal{D}^i)^t \to \mathcal{K} \tag{109}$$

so that in general

$$Z_t = f_t(X^{1,t}, X^{2,t}, \ldots, X^{n,t}) \tag{110}$$

where for all $i = 1, 2, \ldots, n$,

$$X^{i,t} := \left(X_1^i, X_2^i, \ldots X_t^i\right). \tag{111}$$

The signal Z_t is transmitted through n noisy channels to the n receivers. At time t the noise in channel i is described by N_t^i taking values in $\mathcal{Q}^i := \{1, 2, \ldots Q^i\}$. Let $N_t := (N_t^1, N_t^2, \ldots, N_t^n)$. The random variables $N_1, N_2, \ldots, N_T$ are assumed to be mutually independent, and independent of $X_1^i, X_2^i, \ldots, X_T^i, i = 1, 2, \ldots, n$, and each N_t has a known PMF. For each t, N_t^i and $N_t^j, i \neq j$ may be correlated. The signal Y_t^i, received by the ith receiver at time t is a noise-corrupted version of Z_t, that is

$$Y_t^i = h_t^i\left(Z_t, N_t^i\right), \quad i = 1, 2, \ldots, n \tag{112}$$

where h_t^i is a known function that describes channel i at time t, and for each t, Y_t^i takes values in the set $\mathcal{L}^i := \{1, 2, \ldots, L^i\}$.

Receiver $i, i = 1, 2, \ldots, n$, has limited memory. Its memory update is performed as follows.

i) At time $t = 1, Y_1^i$ is available and a discrete random variable

$$W_1^i = l_1^i\left(Y_1^i\right), \quad i = 1, 2, \ldots, n \tag{113}$$

taking values in $\mathcal{M}^i := \{1, 2, \ldots, M^i\}$ is stored in memory. The functions $l_1^i, i = 1, 2, \ldots, n$, are given.

ii) At time $t = 2, 3, \ldots, T$, the memory of receiver $i = 1, 2, \ldots, n$, is updated according to the rule

$$W_t^i = l_t^i\left(W_{t-1}^i, Y_t^i\right) \tag{114}$$

where W_t^i takes values in $\mathcal{M}^i$ and $l_t^i, i = 1, 2, \ldots, n, t = 2, 3, \ldots T$ are given.

At $t = 1, 2, \ldots, T$, receiver i generates a variable $\hat{X}_t^i \in \mathcal{D}^i$ by the rule

$$\hat{X}_1^i = g_1^i\left(Y_1^i\right) \tag{115}$$

$$\hat{X}_t^i = g_t^i\left(Y_t^i, W_{t-1}^i\right) \quad t = 2, \ldots, T \tag{116}$$

where

$$g_1^i : \mathcal{L}^i \to \mathcal{D}^i \tag{117}$$

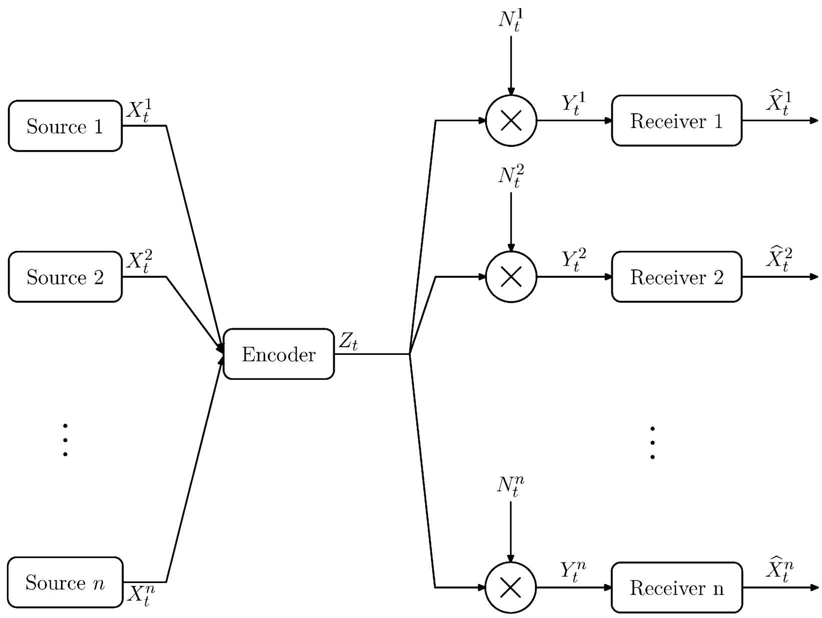

Fig. 2. Broadcast system.

$$g_t^i : \mathcal{L}^i \times \mathcal{M}^i \to \mathcal{D}^i. \tag{118}$$

For each $t = 1, 2, \ldots, T$ and $i = 1, 2, \ldots, n$ the functions

$$\rho_t^i : \mathcal{D}^i \times \mathcal{D}^i \to [0, \infty) \tag{119}$$

are given, and

$$\mathcal{J}_t^i = E\left\{\rho_t^i\left(X_t^i, \hat{X}_t^i\right)\right\} \tag{120}$$

measures the average distortion at receiver i at time t. The system's performance is measured by

$$\begin{aligned}\mathcal{J} &= \sum_{t=1}^{T}\sum_{i=1}^{n}\mathcal{J}_t^i \\ &= \sum_{t=1}^{T}\sum_{i=1}^{n}E\left\{\rho_t^i\left(X_t^i, \hat{X}_t^i\right)\right\}\end{aligned} \tag{121}$$

that is, it is the sum of the distortions of each broadcast transmitter/receiver pair.

The expectation in (121) is with respect to a probability measure that is determined by the distribution of the sequences $X_1^i, X_2^i, \ldots, X_T^i$, the choice of the functions f_t, l_t^i, g_t^i, the channels $h_t^i, i = 1, 2, \ldots, n, t = 1, 2, \ldots, T$, and the statistics of the noise $N_1, N_2, \ldots, N_T$.

The following is assumed.

A1) The statistical description of all the Markov sources $\{X_t^i, t = 1, 2, \ldots, T\}, i = 1, 2, \ldots, n$, is common knowledge ([48], [69]) to the encoder and all the receivers/decoders.

A2) For every $i, i = 1, 2, \ldots, n$, the functions $l_t^i, h_t^i, \rho_t^i, t = 1, 2, \ldots T$, and the statistics of the random process $\{N_1^i, N_2^i, \ldots, N_T^i\}$ are common knowledge to the encoder and receiver i.

Under the above assumptions the optimization problem, Problem (P'), under consideration is the following.

Problem (P'): Consider the above-described model. Given $T, \mathcal{D}^i, \mathcal{K}, \mathcal{L}^i, \mathcal{M}^i, \mathcal{Q}^i,\ h_t^i, \rho_t^i, l_t^i,\ P_{N_t}, t = 1, 2, \ldots, T, i = 1, 2, \ldots, n, P^i_{X_1^i}, P^i_{X_{t+1}^i | X_t^i}, t = 1, 2, \ldots, T-1, i = 1, 2, \ldots n$, choose the functions $f_1, f_2, \ldots, f_T$ and $g_1^i, g_2^i, \ldots, g_T^i, i = 1, 2, \ldots, n$, to minimize $\mathcal{J}$, given by (121).

Because of the real-time constraint on encoding and decoding, the objectives in Problem (P') and the technical approach taken for the solution of Problem (P') are different from those of all previous studies of the broadcast system (see [37]–[44], and the references in [44]).

B. The Structure of Optimal Real-Time Encoders and Decoders

The real-time encoding problem can be viewed as a centralized stochastic control problem where the encoder has to simultaneously control the PMFs of the receivers' memories so as to minimize the performance criterion given by (121) (cf. discussion of Section II-C).

Consider any fixed memory update rules $l^i := (l_1^i, l_2^i, \ldots, l_T^i), i = 1, 2, \ldots, n$, and any arbitrary but fixed decision rules $g^i := (g_1^i, g_2^i, \ldots, g_T^i), i = 1, 2, \ldots, n$ for the decoders. Define for each $t = 1, 2, \ldots, T$

$$X_t := \left(X_t^1, X_t^2, \ldots X_t^n\right) \tag{122}$$

Consider the process $\{S_t, t = 1, 2, \ldots, T\}$ defined by

$$S_1 := X_1 \tag{123}$$

$$S_t := \left(X_t, P_{W_{t-1}^1}, P_{W_{t-1}^2}, \ldots, P_{W_{t-1}^n}\right) \tag{124}$$

where $P_{W_{t-1}^i}, i = 1, 2, \ldots, n$, are defined by (13).

Lemma 5: The process $\{S_t, t = 1, 2, \ldots, T\}$ is conditionally Markov given the Z_t's; that is, for any s^{ot}, z^{ot}

$$P(S_{t+1} | s^{ot}, z^{ot}) = P(S_{t+1} | s_t, z_t) \tag{125}$$

where, for any t

$$s^{ot} := (s_1, s_2, \ldots, s_t) \tag{126}$$

$$z^{ot} := (z_1, z_2, \ldots, z_t). \tag{127}$$

Proof: Define for any t

$$X^{ot} := (X_1, X_2, \ldots, X_t).$$

For any realization $x^{ot}, z^{ot}, \tilde{P}_{W_1^1}, \ldots, \tilde{P}_{W_{t-1}^1}, \ldots, \tilde{P}_{W_1^n}, \ldots, \tilde{P}_{W_{t-1}^n}$ of $X^{ot}, Z^{ot}, P_{W_1^1}, \ldots, P_{W_{t-1}^1}, \ldots, P_{W_1^n}, \ldots, P_{W_{t-1}^n}$, respectively, we have

$$\begin{aligned} P\Big(X_{t+1} = y \,\Big|\, & x^{ot}, z^{ot}, \tilde{P}_{W_1^1}, \ldots, \tilde{P}_{W_{t-1}^1}, \ldots, \\ & \tilde{P}_{W_1^n}, \ldots, \tilde{P}_{W_{t-1}^n}\Big) \\ & = P(X_{t+1} = y \,|\, x_t) \end{aligned} \tag{128}$$

for any $y \in \prod_{i=1}^n \mathcal{D}^i$, by the first-order Markov property of the source. Furthermore, for $t > 1$ and $m_i \in \mathcal{M}^i, i = 1, 2, \ldots, n$,

$$\begin{aligned} P\big(W_t^1 = m_1, \ldots\ldots, W_t^n = m_n \,\big|\, & x^{ot}, z^{ot} \\ & \tilde{P}_{W_1^1}, \ldots, \tilde{P}_{W_{t-1}^1}, \ldots, \tilde{P}_{W_1^n}, \ldots, \tilde{P}_{W_{t-1}^n}\big) \\ = P\big(l_t^1 &\left(h_t^1\left(z_t, N_t^1\right), W_{t-1}^1\right) = m_1, \ldots \\ & \ldots, l_t^n\left(h_t^n\left(z_t, N_t^n\right), W_{t-1}^n\right) \\ = m_n &\,\Big|\, z_t, \tilde{P}_{W_{t-1}^1}, \tilde{P}_{W_{t-1}^2}, \ldots, \tilde{P}_{W_{t-1}^n}\Big) \\ = P((q^1&, q^2, \ldots, q^n, w^1, w^2, \ldots, w^n) \\ \in \prod_{i=1}^n & \mathcal{Q}^i \times \prod_{i=1}^n \mathcal{M}^i \\ : l_t^1 &\left(h_t^1(z_t, q^1), w^1\right) = m_1 \\ \ldots, l_t^n &\left(h_t^n(z_t, q^n), w^n\right) = m_n \\ & \Big| z_t, \tilde{P}_{W_{t-1}^1}, \tilde{P}_{W_{t-1}^2}, \ldots, \tilde{P}_{W_{t-1}^n}\Big) \\ = P\big(W_t^1 &= m_1, W_t^2 = m_2, \ldots, W_t^n = m_n \\ & \Big| z_t, \tilde{P}_{W_{t-1}^1}, \tilde{P}_{W_{t-1}^2}, \ldots, \tilde{P}_{W_{t-1}^n}\Big); \end{aligned} \tag{129}$$

for $t = 1$ and $m_i \in \mathcal{M}^i, i = 1, 2, \ldots, n$

$$\begin{aligned} P\big(W_1^1 = m_1, \ldots\ldots, W_1^n = m_n \,\big|\, x_1, z_1\big) & \\ = P\big(l_1^1\left(h_1^1\left(z_1, N_1^1\right)\right) = m_1, & \\ \ldots, l_1^n\left(h_1^n\left(z_1, N_1^n\right)\right) = m_n \,|\, z_1\big) & \\ = P\big(W_1^1 = m_1, \ldots, W_1^n = m_n \,\big|\, z_1\big). & \end{aligned} \tag{130}$$

The first equality in (129) and the first equality in (130) hold because of (112)–(114) and the assumption that $N_1, N_2, \ldots, N_T$ is a sequence of independent random variables that are also independent of $X_1, X_2, \ldots, X_T$; therefore, for each t, N_t is independent of $X^{ot}, Z^{ot}, W_1^1, \ldots, W_{t-1}^1, \ldots, W_1^n, \ldots, W_{t-1}^n$.

From (129) and (130) we conclude that for $t > 1$

$$P_{W_t^1 W_t^2 \ldots W_t^n} = \mu_t'\left(z_t, \tilde{P}_{W_{t-1}^1}, \tilde{P}_{W_{t-1}^2}, \ldots, \tilde{P}_{W_{t-1}^n}\right) \tag{131}$$

and for $t = 1$

$$P_{W_1^1 W_1^2 \ldots W_1^n} = \mu_1'(z_1) \tag{132}$$

where the joint PMF $P_{W_t^1 W_t^2 \ldots W_t^n}$ (defined by the analogue of (13) for the random vector $W_t^1 W_t^2 \ldots W_t^n$), $t = 1, 2, \ldots, T$, denotes the encoder's perception of the memory of receivers $1, 2, \ldots, n$, at t given z_t and $\tilde{P}_{W_{t-1}^1}, \tilde{P}_{W_{t-1}^2}, \ldots, \tilde{P}_{W_{t-1}^n}$, and $\mu_1', \mu_t', t = 2, 3, \ldots, T$, are functions defined by (130) and (129), respectively, Furthermore, from (131)–(132) we conclude that for $t > 1$

$$\begin{aligned} \Big(P_{W_t^1}, P_{W_t^2}, \ldots, P_{W_t^n}\Big) & \\ = \tilde{\mu}_t & \left(z_t, \tilde{P}_{W_{t-1}^1}, \tilde{P}_{W_{t-1}^2}, \ldots, \tilde{P}_{W_{t-1}^n}\right) \end{aligned} \tag{133}$$

and for $t = 1$

$$\Big(P_{W_1^1}, P_{W_1^2}, \ldots, P_{W_1^n}\Big) = \tilde{\mu}_1(z_1) \tag{134}$$

where $\tilde{\mu}_1, \tilde{\mu}_t, t > 1$, are functions determined by (129)–(132). Consequently, because of (128) and (133) we obtain for $t > 1$ and any $y \in \prod_{i=1}^n \mathcal{D}^i$ and $\bar{P}_{W_t^i} \in \mathcal{P}^{\mathcal{M}^i}, i = 1, 2, \ldots, n$,

$$\begin{aligned} P\Big(S_{t+1} = & \left(y, \bar{P}_{W_t^1}, \ldots, \bar{P}_{W_t^n}\right) \Big|\, s^{ot}, z^{ot}\Big) \\ = P(X_{t+1} & = y \,|\, x_t) \\ \times\, \delta & \Big(\left(\bar{P}_{W_t^1}, \ldots, \bar{P}_{W_t^n}\right), \\ & \tilde{\mu}_t\left(z_t, \tilde{P}_{W_{t-1}^1}, \ldots, \tilde{P}_{W_{t-1}^n}\right)\Big) \\ = P\Big(S_{t+1} & = \left(y, \bar{P}_{W_t^1}, \ldots \bar{P}_{W_t^n}\right) \\ & \Big| x_t, z_t, \tilde{P}_{W_{t-1}^1}, \ldots, \tilde{P}_{W_{t-1}^n}\Big) \\ = P\Big(S_{t+1} & = \left(y, \bar{P}_{W_t^1}, \ldots, \bar{P}_{W_t^n}\right) \,|\, s_t, z_t\Big) \end{aligned} \tag{135}$$

where $\delta(\cdot)$ is the Kronecker delta defined in (75). Equation (135) proves the assertion of Lemma 5. □

The result of Lemma 5 implies that for each t and each realization $x^{ot}, z^{ot}, \tilde{P}_{W_1^1}, \tilde{P}_{W_2^1}, \ldots, \tilde{P}_{W_{t-1}^1}, \ldots, \tilde{P}_{W_1^n}, \ldots, \tilde{P}_{W_{t-1}^n}$, of $X^{ot}, Z^{ot}, P_{W_1^1}, P_{W_2^1}, \ldots, P_{W_{t-1}^1}, \ldots, P_{W_1^n}, \ldots, P_{W_{t-1}^n}$, respectively

$$\begin{aligned} E\Bigg\{\sum_{i=1}^n & \rho_t^i\left(X_t^i, \hat{X}_t^i\right) \Big|\, x^{ot}, z^{ot}, \tilde{P}_{W_1^1}, \ldots, \tilde{P}_{W_{t-1}^1} \\ & \ldots, \tilde{P}_{W_1^n}, \ldots, \tilde{P}_{W_{t-1}^n}\Bigg\} \\ = & \sum_{i=1}^n \sum_{b^i \in \mathcal{D}^i} P\Big(\hat{X}_t^i = b^i \,\Big|\, x^{ot}, z^{ot}, \tilde{P}_{W_1^1}, \ldots, \tilde{P}_{W_{t-1}^1} \\ & \ldots, \tilde{P}_{W_1^n}, \ldots, \tilde{P}_{W_{t-1}^n}\Big)\, \rho_t^i\left(x_t^i, b^i\right) \\ = & \sum_{i=1}^n \sum_{b^i \in \mathcal{D}^i} P\Big((q^i, m^i) \in \mathcal{Q}^i \times \mathcal{M}^i : \\ & : g_t^i(h_t^i(z_t, q^i), m^i) = b^i \,\Big|\, x^{ot}, z^{ot}, \tilde{P}_{W_1^1}, \ldots, \tilde{P}_{W_{t-1}^1}, \\ & \ldots, \tilde{P}_{W_1^n}, \ldots, \tilde{P}_{W_{t-1}^n}\Big)\, \rho_t^i\left(x_t^i, b^i\right) \\ = & \sum_{i=1}^n \sum_{b^i \in \mathcal{D}^i} P\Big((q^i, m^i) \in \mathcal{Q}^i \times \mathcal{M}^i \\ & : g_t^i\left(h_t^i(z_t, q^i), m^i\right) = b^i \,\Big|\, z_{ot}, \tilde{P}_{W_{t-1}^i}\Big)\, \rho_t^i\left(x_t^i, b^i\right) \\ = & \sum_{i=1}^n \bar{\rho}_t^i\left(x_t^i, \tilde{P}_{W_{t-1}^i}, z_t\right) \\ = & \bar{\rho}_t\left(x_t, \tilde{P}_{W_t^1}, \tilde{P}_{W_{t-1}^2}, \ldots, \tilde{P}_{W_{t-1}^n}, z_t\right) \end{aligned} \tag{136}$$

for some functions $\bar{\rho}_t^i, i = 1, 2, \dots n$, and $\bar{\rho}_t$. Consequently, because of (136)

$$\begin{aligned}\mathcal{J} &= E\left\{\sum_{t=1}^{T}\sum_{i=1}^{n}\rho_t^i\left(X_t^i, \hat{X}_t^i\right)\right\} \\ &= E\left\{\sum_{t=1}^{T} E\left\{\sum_{i=1}^{n}\rho_t^i\left(X_t^i, \hat{X}_t^i\right)\middle| X^{ot}, Z^{ot},\right.\right. \\ &\qquad \left.\left. P_{W_1^1}, P_{W_2^1}, \dots, P_{W_{t-1}^1}, \dots, P_{W_1^n}, \dots, P_{W_{t-1}^n}\right\}\right\} \\ &= E\left\{\sum_{t=1}^{T}\bar{\rho}_t\left(X_t, P_{W_{t-1}^1}, P_{W_{t-1}^2}, \dots, P_{W_{t-1}^n}, Z_t\right)\right\} \\ &= E\left\{\sum_{t=1}^{T}\bar{\rho}_t\left(S_t, Z_t\right)\right\}. \end{aligned} \tag{137}$$

Therefore, for fixed $l^1, l^2, \dots l^n$ and $g^1, g^2, \dots g^n$ the problem is to control, through the choice of Z_t, the transition probabilities fromS_t to S_{t+1} so as to minimize the cost given by (137). From Markov decision theory ([36, Ch. 6]) we conclude that an optimal encoding rule is of the form

$$Z_t = f_t\left(X_t, P_{W_{t-1}^1}, P_{W_{t-1}^2}, \dots, P_{W_{t-1}^n}\right). \tag{138}$$

for all $t = 2, 3, \dots T$, and that optimal real-time encoding rules can be determined by the solution of the dynamic program

$$V_{T+1}\left(x^1, x^2, \dots, x^n, \tilde{P}_{W_T^1}, \tilde{P}_{W_T^2}, \dots, \tilde{P}_{W_T^n}\right) = 0 \tag{139}$$

$$\begin{aligned} &V_t\left(x^1, x^2, \dots, x^n, \tilde{P}_{W_{t-1}^1}, \tilde{P}_{W_{t-1}^2}, \dots, \tilde{P}_{W_{t-1}^n}\right) \\ &= \min_{z\in\mathcal{K}}\left\{E_z\sum_{i=1}^{n}\rho_t^i\left(x^i, \hat{X}_t^i\right)\right. \\ &\quad + \sum_{\substack{\left(x_{t+1}^1,\dots,x_{t+1}^n\right)\\ \in\mathcal{D}^1\times\dots\times\mathcal{D}^n}} P\left(x_{t+1}^1, \dots, x_{t+1}^n \,\middle|\, x^1, \dots, x^n\right) \\ &\quad \left.\times V_{t+1}\left(x_{t+1}^1, \dots, x_{t+1}^n, \bar{P}_{W_t^1}^z, \dots, \bar{P}_{W_t^n}^z\right)\right\} \\ &\qquad t = 1, 2, \dots, T \end{aligned} \tag{140}$$

where $\tilde{P}_{W_0}^i := \emptyset, i = 1, 2, \dots, n$, for every $i = 1, 2, \dots, n$

$$\begin{aligned} &E_z\left\{\rho_t^i\left(x^i, \hat{X}_t^i\right)\right\} \\ &= \sum_{b\in\mathcal{D}^i}\rho_t^i(x^i, b)P\left(\hat{X}_t^i = b \,\middle|\, x^i, \tilde{P}_{W_{t-1}^i}, z\right) \\ &= \sum_{b\in\mathcal{D}^i}\rho_t^i(x^i, b)P\left((w^i, n^i)\in\mathcal{M}^i\times\mathcal{Q}^i\right. \\ &\quad \left. : g_t^i\left(h_t^i\left(z, n^i\right), w^i\right) = b \,\middle|\, \tilde{P}_{W_{t-1}^i}, z\right) \end{aligned} \tag{141}$$

and for every $i = 1, 2, \dots, n$, the components of $\bar{P}_{W_t^i}^z$ are given by

$$\begin{aligned}\bar{P}^z\left(W_t^i = c\right) &= P\left((w^i, n^i)\in\mathcal{M}^i\times\mathcal{Q}^i\right. \\ &\quad \left. : l_t^i\left(h_t^i(z, n^i), w^i\right) = c \,\middle|\, z, P_{W_{t-1}^i}\right), \quad c\in\mathcal{M}^i. \end{aligned} \tag{142}$$

We can summarize the results of the above analysis as follows.

Theorem 4: For Problem (P') there is no loss of optimality if one restricts attention to real-time encoders of the form

$$Z_1 = f_1(X_1) = f_1\left(X_1^1, X_1^2, \dots X_1^n\right) \tag{143}$$

$$Z_t = f_t\left(X_t^1, X_t^2, \dots, X_t^n, P_{W_{t-1}^1}, P_{W_{t-1}^2}, \dots, P_{W_{t-1}^n}\right) \quad t = 2, 3, \dots T. \tag{144}$$

Optimal real-time encoding strategies can be determined by the solution of the dynamic program (139)–(142).

The result of Theorem 4 can be intuitively explained as follows. Since the real-time encoder has to produce at each time t one message which it broadcasts to all receivers, it has to take into account the messages produced at t by all the sources, (that is, X_t), and the information it perceives is available to each receiver. This information is described by $P_{W_{t-1}^1}, P_{W_{t-1}^2}, \dots, P_{W_{t-1}^n}$.

The result of Theorem 4 holds for the case where each Markov source is continuous-state discrete-time, and the channel noise is described by a continuous-state random process $N_1, N_2, \dots N_T, (N_t := (N_t^1, N_t^2, \dots, N_t^n))$ for all t, where the random variables $N_t, t = 1, 2, \dots, T$, are mutually independent, each N_t has a known cumulative distribution function and each N_t is independent of $X_1, X_2, \dots, X_T$ (cf. Section III-A). Theorem 4 also holds when the Markov sources $\{X_t^i, t = 1, 2, \dots, T\}, i = 1, 2, \dots, n$, are correlated and the overall process $\{X_t, t = 1, 2, \dots, T\}$ is Markov with a given statistical description. The aforementioned extensions of Theorem 4 can be established by the technical approach presented in this section.

Under Assumptions A1)–A2) (cf. Section IV-A), the real-time decoding problem for each receiver is similar to that in the point-to-point communication system. At each time t, for any fixed $f := (f_1, f_2, \dots, f_T), l^i := (l_1^i, l_2^i, \dots, l_T^i), i = 1, 2, \dots, n$, based on y_t^i, and w_{t-1}^i receiver/decoder i can determine the conditional PMF of X_t^i by a computation similar to that of Appendix IV.

Let

$$\begin{aligned}&\xi_t^{i,f,l^i}(y^i, w^i)(x) \\ &= P^{i,f,l^i}\left(X_t^i = x \,\middle|\, Y_t^i = y^i, W_{t-1}^i = w^i\right) \\ &\qquad x = 1, 2, \dots D^i. \end{aligned} \tag{145}$$

Let $\Xi(\mathcal{D}^i)$ denote the set of PMFs on $\mathcal{D}^i$. For any $\xi\in\Xi(\mathcal{D}^i)$ and $t\geq 1$ define

$$\tau_t^i(\xi) = \arg\min_{b\in\mathcal{D}^i}\sum_{x\in\mathcal{D}^i}\rho_t^i(x, b)\xi(x). \tag{146}$$

Then, by arguing as in the proof of Theorem 2 for each receiver/decoder i we obtain the following result.

Theorem 5: Consider any receiver $i = 1, 2, \dots, n$, and let $f, l^i, i = 1, 2, \dots, n$, be any (fixed) encoding and memory updating strategies, respectively. The optimal real-time decoding rule for receiver i for f, l^i is given by

$$g_1^*\left(Y_1^i\right) = \tau_1\left(\xi_1^{i,f,l^i}\left(Y_1^i\right)\right) \tag{147}$$

$$g_t^*\left(Y_t^i, W_{t-1}^i\right) = \tau_t\left(\xi_t^{i,f,l^i}\left(Y_t^i, W_{t-1}^i\right)\right), \quad t = 2, 3, \dots, T. \tag{148}$$

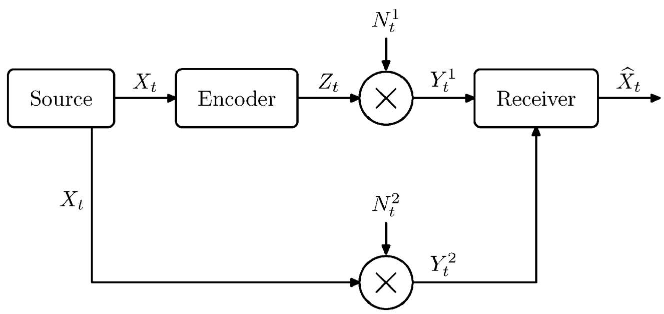

Fig. 3. Variation of Wyner–Ziv problem.

V. A Real-Time Variation of the Wyner–Ziv Problem

A. The Model

Consider the system of Fig. 3. The source is described by a Markov chain $\{X_t, t = 1, 2, \ldots, T\}$ where $X_t \in \mathcal{D}, \mathcal{D}$ is defined in Section II-A1, the PMF P_{X_1} and the transition functions $P_{X_{t+1}|X_t}, t = 1, 2, \ldots, T-1$ are given.

At each time t, a signal Z_t, taking values in the set $\mathcal{K}$, defined in Section II-A1, is transmitted to a receiver. The signal Z_t is produced by a real-time encoder, which is characterized for every t by

$$f_t : \mathcal{D}^t \to \mathcal{K} \tag{149}$$

so that in general

$$Z_t = f_t(X_1, X_2, \ldots, X_t). \tag{150}$$

The signal Z_t is transmitted to the receiver through a noisy channel. At every t, simultaneously with Z_t, the source output X_t is itself transmitted to the same receiver through a second noisy channel. Thus, at each $t = 1, 2, \ldots, T$ the receiver obtains two signals

$$Y_t^1 = h_t^1\left(Z_t, N_t^1\right) \tag{151}$$

and

$$Y_t^2 = h_t^2\left(X_t, N_t^2\right) \tag{152}$$

where $N_t^i, i = 1, 2$, is the noise in channel $i = 1, 2$, and $h_t^i, i = 1, 2$, are known functions describing the two channels at t. Let

$$N_t := \left(N_t^1, N_t^2\right). \tag{153}$$

The random variables $N_1, N_2, \ldots, N_T$ are assumed to be mutually independent, and each N_t is independent of $X_1, X_2, \ldots, X_T$. Furthermore, for each t, N_t^i takes values in $\mathcal{Q}^i := \{1, 2, \ldots, Q^i\}, i = 1, 2$, and Y_t^i takes values in $\mathcal{L}^i := \{1, 2, \ldots, L^i\}, i = 1, 2$.

The receiver has limited memory the update of which is performed as follows:

$$W_1 = l_1\left(Y_1^1, Y_1^2\right) \tag{154}$$
$$W_t = l_t\left(Y_t^1, Y_t^2, W_{t-1}\right) \tag{155}$$

where $l_t, t = 1, 2, \ldots, T$ are given functions. The random variables $W_t, t = 1, 2, \ldots, T$ take values in $\mathcal{M} := \{1, 2, \ldots, M\}$.

At $t = 1, 2, \ldots, T$, the receiver generates an estimate $\hat{X}_t \in \mathcal{D}$ of X_t by the rule

$$\hat{X}_1 = g_1\left(Y_1^1, Y_1^2\right)$$
$$\hat{X}_t = g_t\left(Y_t^1, Y_t^2, W_{t-1}\right), t = 2, 3, \ldots, T \tag{156}$$

where

$$g_1 : \mathcal{L}^1 \times \mathcal{L}^2 \to \mathcal{D} \tag{157}$$

and

$$g_t : \mathcal{L}^1 \times \mathcal{L}^2 \times \mathcal{M} \to \mathcal{D}, \quad t = 2, 3, \ldots, T. \tag{158}$$

For each t a distortion measure $\rho_t(X_t, \hat{X}_t)$ and the average distortion $E\{\rho_t(X_t, \hat{X}_t)\}$ are defined in the same way as in Section II-A2. The system's performance is measured by an index similar to (12), i.e.

$$\mathcal{J} = \sum_{t=1}^{T} \mathcal{J}_t = \sum_{t=1}^{T} E\left\{\rho_t\left(X_t, \hat{X}_t\right)\right\}. \tag{159}$$

The expectation in (159) is with respect to a probability measure that is determined by the distribution of the sequence $X_1, X_2, \ldots, X_T$, the choice of the functions $f := (f_1, f_2, \ldots, f_T), g := (g_1, g_2, \ldots, g_T), l := (l_1, l_2, \ldots, l_T)$, the channels $h_t^1, h_t^2, t = 1, 2, \ldots, T$, and the statistics of the noise $\{N_1^1, N_2^1, \ldots, N_T^1, N_1^2, N_2^2, \ldots, N_T^2\}$.

It is assumed that the model of Section V.A is common knowledge ([48], [69]) to the encoder and the receiver/decoder.

Under this assumption the optimization problem, Problem (P''), for the model described above is the following:

Problem (P'') Given $T, \mathcal{D}, \mathcal{K}, \mathcal{M}, \mathcal{Q}^i, \mathcal{L}^i, i = 1, 2, \rho_t, l_t, h_t^1, h_t^2, P_{N_t^1 N_t^2}, t = 1, 2, \ldots T, P_{X_1}, P_{X_{t+1}|X_t}, t = 1, 2, \ldots, T-1$, choose the functions $f_1, f_2, \ldots, f_T, g_1, g_2, \ldots, g_T$, to minimize $\mathcal{J}$ given by (159).

The above problem is a real-time variation of the Wyner–Ziv problem [34], where in addition to the real-time constraint on encoding and decoding, there is a noisy channel between the encoder and the receiver. Furthermore, in Problem (P'') the source is Markov whereas in [34] the source is described by a sequence of independent identically distributed random variables.

B. The Structure of Optimal Real-Time Encoders and Decoders

By arguments similar to those of Sections II-B, II-D, one can obtain the following results on the structure of optimal real-time encoders and decoders.

Theorem 6: In Problem (P'') there is no loss of optimality if one restricts attention to encoding rules of the form

$$Z_t = f_t\left(X_t, P_{W_{t-1}}\right) \tag{160}$$

for all $t > 1$. Optimal real-time encoding strategies can be determined by the solution of a dynamic program similar to that of (59)–(60).

The real-time decoding problem for the receiver is similar to that in the point-to-point communication system. The following result can be proved in the same way as Theorem 2.

Theorem 7: Let $f := (f_1, f_2, \dots, f_T)$ and $l := (l_1, l_2, \dots, l_T)$ be any fixed encoding and memory updating strategies, respectively. The optimal real-time decoding rule for f, l is given by

$$g_1^*\left(Y_1^1, Y_1^2\right) = \tau_1\left(\xi_1^{f,l}\left(Y_1^1, Y_1^2\right)\right) \tag{161}$$

$$g_t^*\left(Y_t^1, Y_t^2, W_{t-1}\right) = \tau_t\left(\xi_t^{f,l}\left(Y_t^1, Y_t^2, W_{t-1}\right)\right) \quad t = 2, \dots, T \tag{162}$$

where $\xi_t^{f,l}(y^1, y^2, w)$ denotes the conditional PMF of X_t given the decoder's information y^1, y^2, w at time $t, x \in \mathcal{D}$

$$\xi_t^{f,l}(y^1, y^2, w)(x) = P^{f,l}\left(X_t = x \,\middle|\, Y_t^1 = y^1, Y_t^2 = y^2, W_{t-1} = w\right) \tag{163}$$

and $\tau_t(\xi), t = 2, \dots, T$, is defined by (80). The conditional PMFs $\xi_t, t = 2, \dots, T$ can be computed by the method presented in Appendix IV.

The results of Theorems 6 and 7 also hold for models of Markov sources and channels described in Section III-A.

VI. Conclusion

We have discovered the structure of optimal real-time encoders and decoders for point-to-point communication systems, broadcast systems, and a real-time variation of the Wyner–Ziv problem. Our technical approach was based on two key observations. 1) The structure of optimal real-time decoders depends on the distortion measure. 2) For arbitrary but fixed decoding and memory update strategies, optimal real-time encoding is a centralized stochastic control problem where the encoder, through the choice of its strategy, has to optimally control the memory of the receiver(s). Our results imply that the memory size at the encoder's site is independent of T (the finite horizon over which the real-time transmission problem is being considered) and depends only on the size of the memory of the receiver(s). Thus, the optimal real-time encoding problem is substantially simplified when the memory size of the receiver is much smaller than T. Furthermore, optimal real-time encoding strategies can be determined using the computational methods available for the solution of partially observed Markov decision problems. As pointed out in Section II-A3, our results were derived for arbitrary but fixed memory update rule(s). The optimal selection of memory update rule(s), as well as the determination of jointly optimal real-time encoding, decoding and memory update rules, have not been addressed in this paper. A methodology for the determination of jointly optimal real-time encoding, decoding, and memory update strategies appears in [70].

The extension of our results to decentralized real-time encoding-decoding problems that are more general than the Wyner–Ziv model remains an open challenging problem.

Appendix I
Proof of Lemma 1

With a given design $d = (f_1, f_2, l_1, l_2, g_1, g_2)$, we have for any$X_1 = x_1, X_2 = x_2$

$$\begin{aligned}
&E^d\{\rho_2(X_2, \hat{X}_2) | X_1 = x_1, X_2 = x_2\} \\
&= E^d\left\{\rho_2(X_2, \hat{X}_2) \,\middle|\, X_1 = x_1, X_2 = x_2, P^d_{W_1(x_1)}\right\} \\
&= E^d\left\{\rho_2(X_2, g_2(W_1, h_2(Z_2, N_2))) \,\middle|\, X_1 = x_1, X_2 = x_2, P^d_{W_1(x_1)}\right\} \\
&= E^d\left\{\hat{\rho}_2(X_2, W_1, Z_2, N_2) \,\middle|\, X_1 = x_1, X_2 = x_2, P^d_{W_1(x_1)}\right\} \\
&= \sum_{w_1 \in \mathcal{M}} \sum_{z_2 \in \mathcal{K}} \sum_{n_2 \in \mathcal{Q}} P^d\left(W_1 = w_1, Z_2 = z_2, N_2 = n_2 \,\middle|\, X_1 = x_1, X_2 = x_2, P^d_{W_1(x_1)}\right) \hat{\rho}_2(x_2, w_1, z_2, n_2) \\
&= \sum_{z_2 \in \mathcal{K}} P^d(Z_2 = z_2 \,|\, X_1 = x_1, X_2 = x_2) \\
&\quad \times \left[\sum_{n_2 \in \mathcal{Q}} P(N_2 = n_2) \times \sum_{w_1 \in \mathcal{M}} P^d_{W_1(x_1)}(w_1) \hat{\rho}_2(x_2, w_1, z_2, n_2)\right]
\end{aligned} \tag{A1}$$

for some function $\hat{\rho}_2$, where $P^d_{W_1(x_1)}$ is the PMF on the receiver's memory according to the encoder's perception given x_1, (cf. (13))

$$P^d_{W_1(x_1)}(w_1) := P^d(W_1 = w_1 \,|\, X_1 = x_1) \tag{A2}$$

and $P^d_{W_1(x_1)}$ depends on f_1, l_1 but not on f_2. For every $X_1 = x_1, X_2 = x_2$, (A1) quantifies the performance of the design $d := (f_1, f_2, l_1, l_2, g_1, g_2)$ at stage 2, given the information at the encoder's site at stage 2.

Consider now a new design $\hat{d} = (f_1, \hat{f}_2, l_1, l_2, g_1, g_2)$ where

$$\hat{f}_2 : \mathcal{D} \times \mathcal{P}^{\mathcal{M}} \to \mathcal{K} \tag{A3}$$

is chosen as follows. For any given $x_2 \in \mathcal{D}$ and any given $P_{W_1} \in \mathcal{P}^{\mathcal{M}}$

$$\hat{f}_2(x_2, P_{W_1}) = \arg\min_{z_2 \in \mathcal{K}} \left\{\sum_{n_2 \in \mathcal{Q}} P(N_2 = n_2) \times \sum_{w_1 \in \mathcal{M}} P_{W_1}(w_1) \hat{\rho}_2(x_2, w_1, n_2, z_2)\right\}. \tag{A4}$$

Since for some (x_2, P_{w_1}) there may be more than one $z_2 \in \mathcal{K}$ that achieve

$$\min_{z_2 \in \mathcal{K}} \left[\sum_{n_2 \in \mathcal{Q}} P(N_2 = n) \times \left[\sum_{w_1 \in \mathcal{M}} P_{W_1}(w_1) \hat{\rho}_2(x_2, w_1, n_2, z_2)\right]\right]$$

the encoder $\hat{f}_2$ can be constructed by using (A3) and the method proposed in the Appendix of [7]. This method can be briefly described as follows: Consider the set A_1 of information states (x_2, P_{W_1}) for which $z_2 = 1$ is among the minimizing decisions. For all $(x_2, P_{W_1}) \in A_1$, set $\hat{f}_2(x_2, P_{W_1}) = 1$. Next consider the set A_2 of all states for which $z_2 = 2$ is among the minimizing decisions. Let $B_2 = A_2 - A_1$ and for all $(x_2, P_{W_1}) \in B_2$ set $\hat{f}_2(x_2, P_{W_1}) = 2$. Proceed in this way to create the *sets*$B_1 = A_1, \ldots, B_n = A_n - \cup_{j=1}^{n-1} A_j, n = 2, \ldots, \mathcal{K}$, such that if $(x_2, P_{W_1}) \in B_n, \hat{f}_2(x_2, P_{W_1}) = n$. Then, under the design $\hat{d}$ we have for all x_1

$$P^{d}_{W_1(x_1)} = P^{\hat{d}}_{W_1(x_1)} \tag{A5}$$

and

$$\begin{aligned} E^{\hat{d}}&\left\{\rho_2(X_2,\hat{X}_2)\,\Big|\,X_2 = x_2, P^{d}_{W_1(x_1)}\right\} \\ &= E^{\hat{d}}\left\{\rho_2(X_2,\hat{X}_2)\,\Big|\,X_1 = x_1, X_2 = x_2, P^{d}_{W_1(x_1)}\right\} \\ &\leq E^{d}\left\{\rho_2(X_2,\hat{X}_2)\,\Big|\,X_1 = x_1, X_2 = x_2, P^{d}_{W_1(x_1)}\right\} \\ &= E^{d}\left\{\rho_2(X_2,\hat{X}_2)\,|\,X_1 = x_1, X_2 = x_2\right\}. \end{aligned} \tag{A6}$$

Inequality (A6) shows that for a suitable change in $f_2, \mathcal{J}_2$ (given by (11)) can only decrease, that is

$$E^{\hat{d}}(\rho_2(X_2,\hat{X}_2)) \leq E^{d}(\rho_2(X_2,\hat{X}_2)). \tag{A7}$$

Furthermore, since only f_2 is changed

$$E^{\hat{d}}(\rho_1(X_1,\hat{X}_1)) = E^{d}(\rho_1(X_1,\hat{X}_1)). \tag{A8}$$

From (A7) and (A8) we conclude that the design $\hat{d}$ is at least as good as the original design d. This completes the proof of the two-stage lemma.

Remark: The Markov property of the source is not used in the proof of the two-stage lemma, but is needed in the proof of the three-stage lemma.

Appendix II
Proof of Lemma 2

The cost $\mathcal{J}_1^{d'}$ is unaffected by changes in f_2, that is

$$\mathcal{J}_1^{\hat{d}'} = \mathcal{J}_1^{d'} \tag{B1}$$

Furthermore, any changes in f_2 do not affect the receiver's memory W_1

$$W_1 = l_1(Y_1) = l_1(h_1(f_1(X_1), N_1)) = \hat{l}_1(X_1, N_1) \tag{B2}$$

or the PMF $P_{W_1(x_1)}$, (cf. (13))

$$P_{W_1(x_1)}(w_1) := P(W_1 = w_1\,|\,X_1 = x_1) \tag{B3}$$

which depends only on f_1, l_1, and the statistics of the noise N_1.

For any $X_1 = x_1, X_2 = x_2$, the cost incurred at the last two stages by the design d' is

$$\begin{aligned} E^{d'}&\{\rho_2(X_2, g_2(W_1, h_2(Z_2, N_2))) \\ &+ \rho_3(X_3, g_3(W_2, h_3(Z_3, N_3))) \\ &\times |\,X_1 = x_1, X_2 = x_2\} \\ &= \sum_{z_2\in\mathcal{K}}\sum_{n_2\in\mathcal{Q}}\sum_{n_3\in\mathcal{Q}}\sum_{w_2\in\mathcal{M}}\sum_{w_1\in\mathcal{M}}\sum_{z_3\in\mathcal{K}}\sum_{x_3\in\mathcal{D}} \\ &P^{d'}(Z_2 = z_2, N_2 = n_2, N_3 = n_3, W_2 = w_2, W_1 = w_1, \\ &Z_3 = z_3, X_3 = x_3\,|\,X_1 = x_1, X_2 = x_2) \\ &\times [\rho_2(x_2, g_2(w_1, h_2(z_2, n_2))) \\ &+ \rho_3(x_3, g_3(w_2, h_3(z_3, n_3)))] \\ &= \sum_{z_2\in\mathcal{K}} P^{d'}(Z_2 = z_2 | X_1 = x_1 X_2 = x_2) \\ &\times \left[\sum_{n_2\in\mathcal{Q}} P(N_2 = n_2) \times \left[\sum_{n_3\in\mathcal{Q}} P(N_3 = n_3)\right.\right. \\ &\times \left[\sum_{w_2\in\mathcal{M}} P^{d'}(W_2 = w_2\,|\,X_1 = x_1, X_2 = x_2)\right. \\ &\times \left[\sum_{w_1\in\mathcal{M}} P^{d'}(W_1 = w_1\,|\,X_1 = x_1)\right. \\ &\times \left[\sum_{z_3\in\mathcal{K}} P^{d'}(Z_3 = z_3\,|\,X_1 = x_1, X_2 = x_2, X_3 = x_3)\right. \\ &\times \left[\sum_{x_3\in\mathcal{D}} P_2(x_3\,|\,x_2) \times [\rho_2(x_2, g_2(w_1, h_2(z_2, n_2)))\right. \\ &\left.\left.\left.\left.\left.\left. + \rho_3(x_3, g_3(w_2, h_3(z_3, n_3)))]\right]\right]\right]\right]\right]\right]. \end{aligned} \tag{B4}$$

Note that, given d' and any $X_1 = x_1, X_2 = x_2, x_1 \in \mathcal{D}, x_2 \in \mathcal{D}$, one can determine $z_2 = f_2(x_1, x_2)$ and the PMF's $P^{d'}_{W_1(x_1)}, P^{d'}_{W_2(x_1,x_2)}$, where for any $w_1 \in \mathcal{M}$

$$P^{d'}_{W_1(x_1)}(w_1) := P^{d'}(W_1 = w_1\,|\,X_1 = x_1) \tag{B5}$$

and for any $w_2 \in \mathcal{M}$

$$P^{d'}_{W_2(x_1,x_2)}(w_2) := P^{d'}(W_2 = w_2\,|\,X_1 = x_1, X_2 = x_2). \tag{B6}$$

Consequently

$$\begin{aligned} P^{d'}&(Z_3 = z_3\,|\,X_1 = x_1, X_2 = x_2, X_3 = x_3) \\ &= P^{d'}(Z_3 = z_3\,|\,X_1 = x_1, X_2 = x_2, X_3 = x_3 \\ &\quad P^{d'}_{W_2(x_1,x_2)}) \\ &= P^{d'}(Z_3 = z_3\,|\,X_3 = x_3, P^{d'}_{W_2(x_1,x_2)}) \end{aligned} \tag{B7}$$

where the last equality in (B7) follows from the fact that:

$$Z_3 = f_3(X_3, P_{W_2}). \tag{B8}$$

Furthermore,

$$\begin{aligned} P^{d'}&(W_2 = w_2\,|\,X_1 = x_1, X_2 = x_2) \\ &= P^{d'}\Big(W_2 = w_2\,\Big|\,X_1 = x_1, X_2 = x_2, \\ &Z_2 = z_2 = f_2(x_1, x_2), P^{d'}_{W_1(x_1)}\Big) \\ &= P^{d'}\Big(l_2(W_1, h_2(z_2, N_2)) = w_2\,\Big|\,X_1 = x_1, \\ &X_2 = x_2, Z_2 = z_2 = f_2(x_1, x_2), P^{d'}_{W_1(x_1)}\Big) \\ &= P^{d'}\Big(l_2(W_1, h_2(z_2, N_2)) = w_2\,\Big|\,Z_2 = z_2, P^{d'}_{W_1(x_1)}\Big) \\ &= P^{d'}\Big(W_2 = w_2\,\Big|\,z_2, P^{d'}_{W_1(x_1)}\Big). \end{aligned} \tag{B9}$$

Therefore, because of (B7) and (B9), (B4) can be written as

$$\begin{aligned}
&E^{d'}\{\rho_2(X_2,\hat{X}_2)+\rho_3(X_3,\hat{X}_3)\,|\,X_1=x_1,X_2=x_2\}\\
&\quad= E^{d'}\Big\{\rho_2(X_2,\hat{X}_2)+\rho_3(X_3,\hat{X}_3)\\
&\qquad\Big|\,X_1=x_1,X_2=x_2,P^{d'}_{W_1(x_1)}\Big\}\\
&\quad= E^{d'}\Big\{\rho_2(X_2,g_2(W_1,h_2(Z_2,N_2)))\\
&\quad+\rho_3(X_3,g_3(W_2,h_3(Z_3,N_3)))\\
&\qquad\Big|\,X_1=x_1,X_2=x_2,P^{d'}_{W_1(x_1)}\Big\}\\
&\quad= \sum_{z_2\in\mathcal{K}} P^{d'}(Z_2=z_2\,|\,X_1=x_1,X_2=x_2)\\
&\qquad\times\Bigg[\sum_{n_2\in\mathcal{Q}}P(N_2=n_2)\times\Bigg[\sum_{n_3\in\mathcal{Q}}P(N_3=n_3)\\
&\qquad\times\Bigg[\sum_{w_2\in\mathcal{M}}P^{d'}\Big(W_2=w_2\,\Big|\,z_2,P^{d'}_{W_1(x_1)}\Big)\\
&\qquad\times\Bigg[\sum_{w_1\in\mathcal{M}}P^{d'}_{W_1(x_1)}(w_1)\\
&\qquad\times\Bigg[\sum_{z_3\in\mathcal{M}}P^{d'}\Big(Z_3=z_3\,\Big|\,X_3=x_3,P^{d'}_{W_2\left(z_2,P^{d'}_{W_1(x_1)}\right)}\Big)\\
&\qquad\times\Bigg[\sum_{x_3\in\mathcal{D}}P_2(x_3\,|\,x_2)[\rho_2(x_2,g_2(w_1,h_2(z_2,n_2)))\\
&\quad+\rho_3(x_3,g_3(w_2,h_3(z_3,n_3)))]\Bigg]\Bigg]\Bigg]\Bigg]\Bigg]\Bigg]
\end{aligned}\tag{B10}$$

where $P^{d'}_{W_2(z_2,P^{d'}_{W_1(x_1)})}$ denotes the PMF on the receiver's memory at $t=2$ (according to the encoder's perception) given $Z_2=z_2$ and $P^{d'}_{W_1(x_1)}$ (cf. (B5), that is, for any $w_2\in\mathcal{M}$

$$P^{d'}_{W_2\left(z_2,P^{d'}_{W_1(x_1)}\right)}(w_2)=P^{d'}\Big(W_2=w_2\,\Big|\,z_2,P^{d'}_{W_1(x_1)}\Big).\tag{B11}$$

Consider now a new design $\hat{d}' := (f_1,\hat{f}_2,f_3,l_1,l_2,l_3,g_1,g_2,g_3)$ where

$$\hat{f}_2:\mathcal{D}\times\mathcal{P}^{\mathcal{M}}\to\mathcal{K}\tag{B12}$$

is chosen as follows: For any given $x_2\in\mathcal{D}$ and any given $P_{W_1}\in\mathcal{P}^{\mathcal{M}}$

$$\begin{aligned}
z_2=\hat{f}_2(x_2,P_{W_1})&=\arg\min_{z_2\in\mathcal{K}}\Bigg\{\sum_{n_2\in\mathcal{Q}}P(N_2=n_2)\\
&\times\Bigg[\sum_{n_3\in\mathcal{Q}}P(N_3=n_3)\\
&\times\Bigg[\sum_{w_2\in\mathcal{M}}P^{l_2}(W_2=w_2\,|\,z_2,P_{W_1})\\
&\times\Bigg[\sum_{w_1\in\mathcal{M}}P_{W_1}(w_1)\times\Bigg[\sum_{x_3\in\mathcal{D}}P_2(x_3\,|\,x_2)\\
&\times\Bigg[\sum_{z_3\in\mathcal{M}}P^{f_3}\Big(Z_3=z_3\,\Big|\,x_3,P^{l_2}_{W_2\left(z_2,P_{W_1}\right)}\Big)\\
&\times[\rho_2(x_2,g_2(w_1,h_2(z_2,n_2)))\\
&+\ \rho_3(x_3,g_3(w_2,h_3(z_3,n_3)))]\Bigg]\Bigg]\Bigg]\Bigg]\Bigg]\Bigg\}.
\end{aligned}\tag{B13}$$

As in the case of the two-stage lemma, the encoder $\hat{f}_2$ can be constructed by using (B13) and the method described in the Appendix of [7].

Because of the choice of $\hat{d}'$ we have

$$P^{d'}_{W_1(x_1)}(w_1)=P^{\hat{d}'}_{W_1(x_1)}(w_1)\tag{B14}$$

for all $x_1\in\mathcal{D}$ and all $w_1\in\mathcal{M}$, as $P^{d'}_{W_1(x_1)}$ depends only on f_1,h_1,l_1 and the statistics of the noise N_1. Moreover, for any z_2

$$\begin{aligned}
&P^{d'}\Big(W_2=w_2\ |\,z_2,P^{d'}_{W_1(x_1)}\Big)\\
&\qquad=P^{\hat{d}'}\Big(W_2=w_2\,\Big|\,z_2,P^{\hat{d}'}_{W_1(x_1)}\Big)
\end{aligned}\tag{B15}$$

for all $w_2\in\mathcal{M}$, because of (B14) and the fact that for given z_2 and $P_{W_1}, P(W_2=w_2\,|\,z_2,P_{W_1})$ depends only on the channel h_2, the statistics of the noise N_2 and l_2. Finally, for any $z_3\in\mathcal{M}$,

$$\begin{aligned}
&P^{d'}\Big(Z_3=z_3\,\Big|\,x_3,P^{d'}_{W_2\left(z_2,P^{d'}_{W(x_1)}\right)}\Big)\\
&\qquad=P^{\hat{d}'}\Big(Z_3=z_3\,\Big|\,x_3,P^{\hat{d}'}_{W_2\left(z_2,P^{\hat{d}'}_{W_1(x_1)}\right)}\Big)
\end{aligned}\tag{B16}$$

because of (B15) and the fact that the encoding rule f_3 is a separated encoder. As a consequence of (B12)–(B16) we obtain, for any $X_1=x_1,X_2=x_2$,

$$\begin{aligned}
&E^{\hat{d}'}\Big\{\rho_2(X_2,\hat{X}_2)+\rho_3(X_3,\hat{X}_3)\,\Big|\,X_2=x_2,P^{\hat{d}'}_{W_1(x_1)}\Big\}\\
&\quad=E^{\hat{d}'}\Big\{\rho_2(X_2,\hat{X}_2)+\rho_3(X_3,\hat{X}_3)\\
&\qquad\Big|\,X_1=x_1,X_2=x_2,P^{\hat{d}'}_{W_1(x_1)}\Big\}\\
&\quad\le E^{d'}\Big\{\rho_2(X_2,\hat{X}_2)+\rho_3(X_3,\hat{X}_3)\\
&\qquad\Big|\,X_1=x_1,X_2=x_2,P^{d'}_{W_1(x_1)}\Big\}\\
&\quad=E^{d'}\Big\{\rho_2(X_2,\hat{X}_2)+\rho_3(X_3,\hat{X}_3)\\
&\qquad\Big|\,X_1=x_1,X_2=x_2\Big\}
\end{aligned}\tag{B17}$$

therefore

$$\mathcal{J}_2^{\hat{d}'}+\mathcal{J}_3^{\hat{d}'}\le\mathcal{J}_2^{d'}+\mathcal{J}_3^{d'}.\tag{B18}$$

Furthermore, from (B1) we have

$$\mathcal{J}_1^{\hat{d}'}=\mathcal{J}_1^{d'}\tag{B19}$$

so that

$$\begin{aligned}
\mathcal{J}^{\hat{d}'}&=\mathcal{J}_1^{\hat{d}'}+\mathcal{J}_2^{\hat{d}'}+\mathcal{J}_3^{\hat{d}'}\\
&\le\mathcal{J}_1^{d'}+\mathcal{J}_2^{d'}+\mathcal{J}_3^{d'}=\mathcal{J}^{d'}.
\end{aligned}\tag{B20}$$

Inequality (B20) shows that for a suitable change of f_2 by a separated encoder $\hat{f}_2$ the overall cost can only decrease. This completes the proof of the three-stage lemma.

Remarks:

1) The Markovian nature of the source is used in the proof of the three-stage lemma, specifically, in establishing the second equality in (B4).
2) In (B10) the terms $P^{d'}(W_2 = w_2 \,|\, z_2, P^{d'}_{W_1(x_1)}), w_2 \in \mathcal{M}$ depend only on l_2, the channel $h_2(\cdot)$, and the statistics of the noise N_2. The PMF $P^{d'}_{W_2(z_2, P^{d'}_{W_1(x_1)})}$ changes as z_2 and/or $P^{d'}_{W_1(x_1)}$ vary. Also, in (B10) the terms $P^{d'}(Z_3 = z_3 \,|\, X_3 = x_3, P^{d'}_{W_2(z_2, P^{d'}_{W_1(x_1)})}), z_3 \in \mathcal{K}$, depend only on f_3; the values of these probabilities change as x_3 and/or $P^{d'}_{W_2(z_2, P^{d'}_{W_1(x_1)})}$ vary.

APPENDIX III
PROOF OF LEMMA 3

The given T-stage system can be considered as a two-stage system by setting

$$\bar{X}_1 := (X_1, X_2, \ldots, X_{T-1}), \tag{C1}$$
$$\bar{X}_2 := X_T, \tag{C2}$$
$$\bar{N}_1 := (N_1, N_2, \ldots, N_{T-1}), \tag{C3}$$
$$\bar{N}_2 := N_T, \tag{C4}$$
$$\bar{Z}_1 := (Z_1, Z_2, \ldots, Z_{T-1}), \tag{C5}$$
$$\bar{Z}_2 := Z_T, \tag{C6}$$
$$\bar{Y}_1 := (Y_1, Y_2, \ldots, Y_{T-1}), \tag{C7}$$
$$\bar{Y}_2 := Y_T, \tag{C8}$$
$$\bar{W}_1 := W_{T-1} = \phi(\bar{X}_1, \bar{N}_1), \tag{C9}$$

where ϕ is defined in terms of $f_1, f_2, \ldots, f_{T-1}, l_1, l_2, \ldots, l_{T-1}, h_1, h_2, \ldots, h_{T-1}$,

$$\hat{\bar{X}}_1 := (\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_{T-1}), \tag{C10}$$
$$\hat{\bar{X}}_2, := \hat{X}_T, \tag{C11}$$
$$\bar{f}_2(\bar{X}_1, \bar{X}_2) := f_T(X_1, X_2, \ldots, X_{T-1}, X_T), \tag{C12}$$
$$\bar{l}_2(\bar{W}_1, \bar{Y}_2) := l_T(W_{T-1}, Y_T), \tag{C13}$$
$$\bar{g}_2(\bar{W}_1, \bar{Y}_2) := g_T(W_{T-1}, Y_T), \tag{C14}$$
$$\bar{\rho}_1(\bar{X}_1, \hat{\bar{X}}_1) := \sum_{t=1}^{T-1} \rho_t(X_t, \hat{X}_t), \tag{C15}$$
$$\bar{\rho}_2(\bar{X}_2, \hat{\bar{X}}_2) := \rho_T(X_T, \hat{X}_T), \tag{C16}$$
$$\bar{\mathcal{J}}_1^d := E^d\{\bar{\rho}_1(\bar{X}_1, \hat{\bar{X}}_1)\} = E^d\left\{\sum_{t=1}^{T-1} \rho_t(X_t, \hat{X}_t)\right\}, \tag{C17}$$
$$\bar{\mathcal{J}}_2^d := E^d\{\bar{\rho}_2(\bar{X}_2, \hat{\bar{X}}_2)\} = E^d\left\{\rho_T(X_T, \hat{X}_T)\right\}. \tag{C18}$$

Then, by the two-stage lemma there is an encoder $\hat{f}_2$ that has the structure

$$\bar{Z}_2 = \hat{f}_2(\bar{X}_2, P_{\bar{W}_1}) \tag{C19}$$

and is such that its use does not increase the cost $\bar{\mathcal{J}}_2$. In the original notation, this corresponds to an encoder $\hat{f}_T$ that has the structure

$$Z_T = \hat{f}_T(X_T, P_{W_{T-1}}) \tag{C20}$$

and the use of which does not increase the cost $\mathcal{J}_T$. Since $\bar{\mathcal{J}}_1 = \sum_{t=1}^{T-1} \mathcal{J}_t$ remains unchanged when f_T is replaced by $\hat{f}_T$, the overall cost $\mathcal{J} = \sum_{t=1}^{T} \mathcal{J}_t$ does not increase by the use of $\hat{f}_T$, and this completes the proof of Lemma 3.

APPENDIX IV

The random functions $\xi_1^{f,l}(Y_1)$ and $\xi_t^{f,l}(Y_t, W_{t-1}), t = 2, 3, \ldots, T$ can be computed as follows. For $t = 1$ and any $y \in \mathcal{L}, x \in \mathcal{D}$

$$\xi_1^{f,l}(y)(x) = \frac{P^{f,l}(Y_1 = y \,|\, x) P_{X_1}(x)}{\sum_{x' \in \mathcal{D}} P^{f,l}(Y_1 = y \,|\, x') P_{X_1}(x')}. \tag{D1}$$

Since

$$Y_1 = h_1(Z_1, N_1) = h_1(f_1(X_1), N_1) \tag{D2}$$

it follows from (D1) and (D2) that, for any $y \in \mathcal{L}, x \in \mathcal{D}$

$$P^{f,l}(Y_1 = y \,|\, x) = P(n \in Q : h_1(f_1(x, n)) = y). \tag{D2}$$

Then, for any $y \in \mathcal{L}, x \in \mathcal{D}, \xi_1^{f,l}(y)(x)$ is determined by (D1) and (D3). For any $t > 1$ and $y \in \mathcal{L}, w \in \mathcal{M}, x \in \mathcal{D}$

$$\xi_t(y, w)(x) = \frac{P(Y_t = y, W_{t-1} = w \,|\, x) P_{X_t}(x)}{\sum_{x' \in \mathcal{D}} P(Y_t = y, W_{t-1} = w \,|\, x') P_{X_t}(x')}. \tag{D4}$$

Furthermore, for any t, because of (1), (4), and (5), we obtain

$$\begin{aligned} Y_t &= h_t(Z_t, N_t) = h_t(f_t(X_1, X_2, \ldots, X_t), N_t) \\ &= \tilde{h}_t(X_1, X_2, \ldots, X_t, N_t) \end{aligned} \tag{D5}$$

for some function $\tilde{h}_t$, and

$$\begin{aligned} W_t &= l_t(Y_t, W_{t-1}) = l_t(Y_t, l_{t-1}(Y_{t-1}, W_{t-2})) \\ &= \ldots = \hat{l}_t(Y_t, Y_{t-1}, \ldots, Y_1) \\ &= \tilde{l}_t(X_1, X_2, \ldots, X_t, N_1, N_2, \ldots, N_t) \end{aligned} \tag{D6}$$

for some functions $\hat{l}_t$ and $\tilde{l}_t$. Using (D5) and (D6) we can write

$$\begin{aligned} &P(Y_t = y, W_{t-1} = w \,|\, x) \\ &= \sum_{x_1 \in \mathcal{D}} \sum_{x_2 \in \mathcal{D}} \cdots \sum_{x_{t-1} \in D} P(x_1, x_2, \ldots, x_{t-1} \,|\, x) \\ &\quad \times P(Y_t = y, W_{t-1} = w \,|\, x_1, x_2, \ldots, x_{t-1}, x) \\ &= \sum_{x_1 \in \mathcal{D}} \sum_{x_2 \in \mathcal{D}} \cdots \sum_{x_{t-1} \in \mathcal{D}} P(x_1, x_2, \ldots, x_{t-1} \,|\, x) \\ &\quad \times P(\tilde{h}_t(X_1, X_2, \ldots, X_t, N_t) = y, \\ &\tilde{l}_{t-1}(X_1, X_2, \ldots, X_{t-1}, N_1, N_2, \ldots, N_{t-1}) = w \end{aligned}$$

$$\begin{aligned}&\quad | x_1, x_2, \ldots, x_{t-1}, x) \\ &= \sum_{x_1 \in \mathcal{D}} \sum_{x_2 \in \mathcal{D}} \cdots \sum_{x_{t-1} \in \mathcal{D}} P(x_1, x_2, \ldots, x_{t-1} \,|\, x) \\ &\quad \times P((n_1, n_2, \ldots, n_t) \in \mathcal{Q}^t : \\ &\tilde{h}_t(x_1, x_2, \ldots, x_{t-1}, x, n_t) = y, \\ &\tilde{l}_{t-1}(x_1, x_2, \ldots, x_{t-1}, n_1, n_2, \ldots, n_{t-1}) = w) \\ &= \sum_{x_1 \in \mathcal{D}} \sum_{x_2 \in \mathcal{D}} \cdots \sum_{x_{t-1} \in \mathcal{D}} P(x_1, x_2, \ldots, x_{t-1} \,|\, x) \\ &\quad \times P(n_t \in \mathcal{Q} : \tilde{h}_t(x_1, x_2, \ldots, x_{t-1}, x, n) = y) \\ &\quad \times P((n_1, n_2, \ldots, n_{t-1}) \in \mathcal{Q}^{t-1}: \\ &\tilde{l}_t(x_1, x_2, \ldots, x_{t-1}, n_1, n_2, \ldots, n_{t-1}) = w) \qquad \text{(D7)}\end{aligned}$$

where the third and fourth equalities in (D7) follow from the fact that the random variable $N_1, N_2, \ldots, N_T$ are mutually independent and independent of $X_1, X_2, \ldots, X_T$. The probability $P(x_1, x_2, \ldots, x_{t-1} \,|\, x)$ can be computed using the PMF P_{X_1} and the transition probabilities $P_{X_{s+1} \,|\, X_s}$, $s = 1, 2, \ldots, t-1$. Moreover

$$\begin{aligned}&P((n_1, n_2, \ldots, n_{t-1}) \in \mathcal{Q}^{t-1}: \\ &\quad \tilde{l}_{t-1}(x_1, x_2, \ldots, x_{t-1}, n_1, n_2, \ldots, n_{t-1}) = w) \\ &\quad = \sum_{(n_1, n_2, \ldots, n_{t-1}) \in A(x_1, x_2, \ldots, x_{t-1})} P(n_1, n_2, \ldots, n_t) \\ &\qquad \text{(D8)}\end{aligned}$$

where

$$\begin{aligned}A(x_1, x_2, \ldots, x_{t-1}) &:= \{(n_1, n_2, \ldots, n_{t-1}) \in \mathcal{Q}^{t-1}: \\ &\tilde{l}_{t-1}(x_1, x_2, \ldots, x_{t-1}, n_1, n_2, \ldots, n_{t-1}) = w\} \qquad \text{(D9)}\end{aligned}$$

and each of the terms in the sum of the right-hand side of (D8) can be computed using the mutual independence of the random variables $N_1, N_2, \ldots, N_T$.

Then for any f, l and any $y \in \mathcal{L}, w \in \mathcal{M}, x \in \mathcal{D}, \xi_t^{f,l}(y, w(x)$ can be computed using (D4), (D7), (D8) and (D9).

Acknowledgment

The author is indebted to A. Anastasopoulos, S. Baveja, A. Mahajan, D. Neuhoff, S. Pradhan, and S. Savari for stimulating discussions. He is also grateful to the anonymous reviewers whose comments significantly improved the presentation of the paper.

References

[1] R. G. Gallager, *Information Theory and Reliable Communication*. New York: Wiley, 1968.
[2] J. C. Walrand and P. Varaiya, "Optimal causal coding—decoding problems," *IEEE Trans. Inf. Theory*, vol. IT-29, no. 6, pp. 814–820, Nov. 1983.
[3] T. Fine, "Properties of an optimum digital system and applications," *IEEE Trans. Inf. Theory*, vol. IT-10, pp. 287–296, Oct. 1964.
[4] D. L. Neuhoff and R. K. Gilbert, "Causal source codes," *IEEE Trans. Inf. Theory*, vol. IT-28, no. 6, pp. 701–713, 1982.
[5] T. Linder and R. Zamir, "Causal source coding of stationary sources with high resolution," in *Proc. ISIT 2001*, Washington, D.C.
[6] P. Piret, "Causal sliding block encoders with feedback," *IEEE Trans. Inf. Theory*, vol. IT-25, no. 2, pp. 237–240, Mar. 1979.
[7] H. S. Witsenhausen, "On the structure of real-time source coders," *Bell Syst. Tech. J.*, vol. 58, no. 6, pp. 1437–1451, Jul.–Aug. 1978.
[8] J. G. Dunham, Optimal Source Coders with Real-Time Constraints, preprint.
[9] T. Linder and G. Lugosi, "A zero-delay sequential quantizer for individual sequences," *IEEE Trans. Inf. Theory*, vol. 47, no. 6, pp. 2533–2538, Sep. 2001.
[10] V. S. Borkar, S. K. Mitter, and S. Tatikonda, "Optimal sequential vector quantization of Markov sources," *SIAM J. Contr. Optim.*, vol. 40, no. 1, pp. 135–148, Jan. 2001.
[11] N. T. Gaarder and D. Slepian, "On optimal finite-state digital transmission systems," *IEEE Trans. Inf. Theory*, vol. IT-28, no. 2, pp. 167–186, Mar. 1982.
[12] J. G. Gibson, T. R. Fischer, and B. Koo, "Preposterior analysis for differential encoder design," *IEEE Trans. Inf. Theory*, vol. IT-32, no. 3, pp. 375–382, May 1986.
[13] S. P. Lloyd, "Bicausal isomorphism of Bernoulli processes," *Bell Lab. Tech. Memo. 75-1217-16*, Nov. 1975.
[14] G. Szekeres, G. Gabor, and Z. Győfri, On the Gaarder–Slepian tracking system conjecture, Department of Mathematics, Statistics, and Computer Science, Tech. Rep., Dalhousie University Halifax, N.S., Canada, 1987.
[15] B. McMillan, "Communication systems which minimize coding noise," *Bell Syst. Tech. J.*, vol. 48, no. 9, pp. 3091–3113, Sep. 1969.
[16] A. W. Drake, "Observation of a Markov process through a noisy channel," Sc.D. Thesis, Department of Electrical Engineering, MIT, Cambridge, MA, 1962.
[17] J. L. Devore, "A note on the observation of a Markov source through a noisy channel," *IEEE Trans. Inf. Theory*, vol. IT-20, pp. 762–764, 1974.
[18] R. S. Liptser and A. N. Shiryayev, *Statistics of Random Processes*. New York, NY: Springer-Verlag, 1977, vol. II: Applications.
[19] S. Tatikonda, "Control Under Communication Constraints," Ph.D. dissertation, Department of EECS, MIT, Cambridge, MA, 2000.
[20] T. Basar and R. Bansal, "Optimum design of measurement channels and control policies for linear-quadratic stochastic systems," *European J. Oper. Res.*, vol. 93, pp. 226–236, 1994.
[21] R. Bansal and T. Basar, "Simultaneous design of measurement and control strategies for stochastic systems with feedback," *Automatica*, vol. 25, no. 5, pp. 679–694, 1989.
[22] H. S. Witsenhausen, "Informational aspects of stochastic control," in *Proc. Oxford Conf. Stoch. Optim.*, 1978.
[23] D. Teneketzis, "Communication in Decentralized Control," Ph.D. dissertation, Department of EECS, MIT, Cambridge, MA, Sep. 1979.
[24] G. Munson, "Causal Information Transmission with Feedback," Ph.D. dissertation, School of Electrical Engineering, Cornell University, Ithaca, NY, 1981.
[25] A. K. Gorbunov and M. S. Pinsker, "Non-anticipatory and prognostic epsilon entropies and message generation rates," *Probl. Inf. Trans.*, vol. 9, pp. 184–191, 1973.
[26] A. K. Gorbunov and M. S. Pinsker, "Prognostic epsilon entropy of a Gaussian message and a Gaussian source," *Probl. Inf. Transm.*, vol. 10, pp. 99–109, 1974.
[27] M. S. Pinsker and A. K. Gorbunov, "Epsilon entropy with delay with small mean-square reproduction error," *Probl. Inf. Transm.*, vol. 23, no. 2, pp. 3–8, Apr.–Jun. 1987.
[28] S. Engell, "New results on the real-time transmission problem," *IEEE Trans. Inf. Theory*, vol. IT-33, no. 2, Mar. 1987.
[29] Y. C. Ho, M. Kastner, and E. Wong, "Team signaling and information theory," *IEEE Trans. Autom. Control*, vol. AC-23, no. 2, pp. 305–312, Apr. 1978.
[30] T. Ericson, Delayless Information Transmission, Linköping University, Linköping, Sweden, LiTH-ISY-I-0260, 1979.
[31] J. C. Kieffer and J. G. Dunham, "On a type of stochastic stability for a class of encoding schemes," *IEEE Trans. Inf. Theory*, vol. IT-29, no. 6, pp. 793–797, Nov. 1983.
[32] J. C. Kieffer, "Stochastic stability for feedback quantization schemes," *IEEE Trans. Inf. Theory*, vol. IT-28, no. 2, pp. 248–254, Mar. 1982.
[33] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, ser. Telecommunications. New York: Wiley, 1991.
[34] A. D. Wyner and J. Ziv, "The rate distortion function for source coding with side information at the decoder," *IEEE Trans. Inf. Theory*, vol. IT-22, pp. 1–10, 1976.
[35] R. D. Smallwood and E. J. Sondik, "The optimal control of partially observable Markov processes over a finite horizon," *Oper. Res.*, vol. 11, pp. 1071–1088, 1973.
[36] P. R. Kumar and P. Varaiya, *Stochastic Systems: Estimation, Identification, and Adaptive Control*. Englewood Cliffs, NJ: Prentice Hall, 1986.

[37] T. M. Cover, "Broadcast channels," *IEEE Trans. Inf. Theory*, vol. IT-18, pp. 2–14, Jan. 1972.
[38] R. G. Gallager, "Capacity and coding for degraded broadcast channels," *Probl. Inform. Transm.*, pp. 185–193, July–Sept. 1974.
[39] S. I. Gelfand and M. S. Pinsker, "Capacity of a broadcast channel with one deterministic component," *Probl. Inf. Transm.*, vol. 16, no. 1, pp. 17–25, Jan.–Mar 1980.
[40] S. I. Gelfand and M. S. Pinsker, "Coding for channel with random parameters," *Probl. Contr. Inf. Theory*, vol. 9, no. 1, pp. 19–31, 1980.
[41] K. Marton, "A coding theorem for the discrete memoryless broadcast channel," *IEEE Trans. Inf. Theory*, vol. IT-25, pp. 306–311, May 1979.
[42] G. Caire and S. Shamai, "On the achievable throughput of a multi-antenna Gaussian broadcast channel," *IEEE Trans. Inf. Theory*, vol. 49, no. 7, pp. 1691–1706, Jul. 2003.
[43] S. Viswanath, N. Jindal, and A. Goldsmith, "Duality, achievable rates, and sum-rate capacity of gausian MIMO broadcast channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2658–2668, Oct. 2003.
[44] T. Cover, "Comments on broadcast channels," *IEEE Trans. Inf. Theory*, vol. IT-44, no. 6, pp. 2524–2531, Oct. 1998.
[45] S. P. Lloyd, "Rate versus fidelity for the binary source," *Bell Syst. Tech. J.*, vol. 56, pp. 427–437, Mar. 1977.
[46] A. Sahai, "Anytime Information Theory," Ph.D. dissertation, Dept. EECS, MIT, Cambridge, MA, Feb. 2001.
[47] W. S. Lovejoy, "A survey of algorithmic methods for partially observed Markov decision processes," *Ann. Oper. Res.*, vol. 28, 1991.
[48] R. J. Aumann, "Agreeing to disagree," *Ann. Stat.*, vol. 4, no. 6, pp. 1236–1239.
[49] C. C. White, "Partially observed Markov decision processes: A survey," *Ann. Oper. Res.*, vol. 32.
[50] T. Weissman and N. Merhav, "Limited-delay lossy coding for filtering of individual sequences," *IEEE Trans. Inf. Theory*, vol. IT-48, no. 3, pp. 721–733, Mar. 2002.
[51] A. Gyorgi, T. Linder, and G. Lugosi, "A follow the perturbed leader-type algorithm for zero-delay quantization of individual sequences," in *Data Compression Conference*, Snowbird, UT, Mar. 23–25, 2004.
[52] A. Gyorgi, T. Linder, and G. Lugosi, "Efficient adaptive algorithms and minimax bounds for zero-delay lossy source coding," *IEEE Trans. Inf. Theory*, vol. 52, pp. 2337–2347, Aug. 2004.
[53] N. Merhav and I. Kontoyannis, "Source coding exponents for zero-delay coding with finite memory," *IEEE Trans. Inf. Theory*, vol. IT-49, no. 3, pp. 609–625, Mar. 2003.
[54] S. Matloub and T. Weismann, "On competitive zero-delay source-channel coding," in *Proc. CISS Conf.*, Princeton, N.J., Mar. 17–19, 2004.
[55] W. Zhang, "Algorithms for Partially Observable Markov Decision Processes," Ph.D. dissertation, The Hong Kong University of Science and Technology, , Aug. 2001.
[56] R. Jain and P. Varaiya, "Simulation-based uniform value function estimates of Markov decision processes," *SIAM J. Contr. Optim.*, Nov. 2004.
[57] R. Jain, "Efficient Market Mechanisms and Simulation-based Learning for Multi-agent Systems," Ph.D. dissertation, Dep. EECS, , University of California, Berkeley, Berkeley, CA, 2004.
[58] W. S. Lovejoy, "Computationally feasible bounds for partially observed Markov decision processes," *Oper. Res.*, vol. 39, no. 1, pp. 162–175, 1991.
[59] E. J. Sondik, "The Optimal Control of Partially Observed Markov Decision Processes," Ph.D. Dissertation, Stanford University, Stanford, CA, 1971.
[60] E. J. Sondik, "The optimal control of partially observable decision processes over the infinite horizon: Discounted cost," *Oper. Res.*, vol. 26, no. 2, pp. 282–304, 1978.
[61] C. C. White III and W. T. Scherer, "Solution procedures for partially observed Markov decision processes," *Oper. Res.*, vol. 37, no. 5, pp. 791–797, 1989.
[62] J. Rust, "Structural estimation of Markov decision processes," in *Handbook of Econometrics*, R. F. Engle and D. L. McFadden, Eds. : Elsevier, 1994, vol. IV, ch. 51, pp. 3081–3143.
[63] J. Rust, "Numerical Dynamic Programming in Economics," in *Handbook of Computational Economics*, H. M. Amman, D. A. Kendrick, and J. Rust, Eds. : Elsevier, 1996, vol. I, ch. 14, pp. 619–729.
[64] D. P. Bertsekas and J. N. Tsitsiklis*, Neuro-Dynamic Programming*. Boston, MA: Athena Scientific, 1996.
[65] J. Pineau, G. Gordon, and S. Thrun, "Point-based value iteration: An anytime algorithm for POMDPs," in *Proc. Int. Joint Conf. Artif. Intell. (IJCAI)*, Acapulco, Mexico, 2003, pp. 1025–1032.
[66] A. Y. Ng and M. Jordan, "PEGASUS: A policy search method for large MDP's and POMDPs," *Proc. Uncertain. Artif. Intell.*, 2000.
[67] P. Pompart and C. Boutilier, "Value-directed compression for POMDPs," in *Advances in Neural Information Processing Systems*. Cambridge, MA: MIT Press, 2003, vol. 15.
[68] T. Linder and R. Zamir, "Causal source coding of stationary sources with high resolution," in *Proc. Int. Symp. Inf. Theory*, Washington, DC, Jun. 2001, p. 28.
[69] R. B. Washburn and D. Teneketzis, "Asymptotic agreement among communicating decisionmakers," *Stochastics*, vol. 13, pp. 103–129, 1984.
[70] A. Mahajan and D. Teneketzis, "A decision theoretic framework for real-time communication," *Proc. 43rd Allerton Conf. Contr., Commun. Comput.*, Sep. 2005.
[71] T. Ericson, "A result on delay-less information transmission," in *Proc. Int. Symp. Inf. Theory*, Grignano, Italy, Jun. 1979.