Local Public Good Provisioning in Networks: A Nash Implementation Mechanism

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***Abstract*—In this paper we study resource allocation in decentralized information local public good networks. A network is a local public good network if each user's actions directly affect the utility of an arbitrary subset of network users. We consider networks where each user knows only that part of the network that either affects or is affected by it. Furthermore, each user's utility and action space are its private information, and each user is a self utility maximizer. This network model is motivated by several applications including wireless communications. For this network model we formulate a decentralized resource allocation problem and develop a decentralized resource allocation mechanism (game form) that possesses the following properties: (i) All Nash equilibria of the game induced by the mechanism result in allocations that are optimal solutions of the corresponding centralized resource allocation problem (Nash implementation). (ii) All users voluntarily participate in the allocation process specified by the mechanism (individual rationality). (iii) The mechanism results in budget balance at all Nash equilibria and off equilibrium.**

***Index Terms*—Network, local public good, decentralized resource allocation, mechanism design, Nash implementation, budget balance, individual rationality.**

I. Introduction

IN NETWORKS individuals' actions often influence the performance of their directly connected neighbors. Such an influence of individuals' actions on their neighbors' performance can propagate through the network affecting the performance of the entire network. Examples include several real world networks; e.g. in a wireless cellular network, the transmission of the base station to a given user (an action corresponding to this user) creates interference to the reception of other users and affects their performance. In an urban network, when a jurisdiction institutes a pollution abatement program, the benefits also accrue to nearby communities. The influence of neighbors is also observed in the spread of information and innovation in social and research networks. Networks with above characteristics are called local public good networks.

A local public good network differs from a typical public good system in that a local public good (alternatively, the action of an individual) is accessible to and directly influences the utilities of individuals in a particular neighborhood within a big network. On the other hand a public good is accessible to and directly influences the utilities of all individuals in the system ([1, Chapter 11]). Because of the localized interactions of individuals, in local public good networks (such as ones described above) the information about the network is often localized; i.e., the individuals are usually aware of only their neighborhoods and not the entire network. In many situations the individuals also have some private information about the network or their own characteristics that are not known to anybody else in the network. Furthermore, the individuals may also be selfish who care only about their own benefit in the network. Such a decentralized information local public good network with selfish users gives rise to several research issues. In the next section we provide a literature survey on prior research in local public good networks.

A. Literature survey

There exists a large literature on local public goods within the context of local public good provisioning by various municipalities that follows the seminal work of [2]. These works mainly consider network formation problems in which individuals choose where to locate based on their knowledge of the revenue and expenditure patterns (on local public goods) of various municipalities. In this paper we consider the problem of determining the levels of local public goods (actions of network agents) for a given network; thus, the problem addressed in this paper is distinctly different from those in the above literature. Recently, Bramoullé and Kranton [3] and Yuan [4] analyzed the influence of selfish users' behavior on the provision of local public goods in networks with fixed links among the users. The authors of [3] study a network model in which each user's payoff equals its benefit from the sum of efforts (treated as local public goods) of its neighbors less a cost for exerting its own effort. For concave benefit and linear costs, the authors analyze Nash equilibria (NE) of the game where each user's strategy is to choose its effort level that maximizes its own payoff from the provisioning of local public goods. The authors show that at such NE *specialization* can occur, i.e. only a subset of individuals contribute to the local public goods and others free ride. In [4] the work of [3] is extended to directed networks where the externality effects of users' efforts on each others' payoffs can be unidirectional or bidirectional. The authors of [4] investigate the relation between the structure of directed networks and the existence and nature of Nash equilibria of users' effort levels in those networks. However, it is shown in [3], [4] that none of the NE of the abovementioned games result in a local public goods provisioning that achieves optimum social welfare.

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In this paper we consider a generalization of the network models investigated in [3], [4]. Specifically, we consider a fixed network where the actions of each user directly affect the utilities of an arbitrary subset of network users. In our model, each user's utility from its neighbors' actions is an arbitrary concave function of its neighbors' action profile. Each user in our model knows only that part of the network that either affects or is affected by it. Furthermore, each user's utility and action space are its private information, and each user is a self utility maximizer. Even though the network model we consider has similarities with those investigated in [3], [4], the problem of local public goods provisioning we formulate in this paper is different from those in [3], [4]. Specifically, we formulate a problem of local public goods provisioning in the framework of implementation theory [1] and address questions such as – How should the network users communicate so as to preserve their private information, yet make it possible to determine actions that achieve optimum social welfare? How to provide incentives to the selfish users to take actions that optimize the social welfare? How to make the selfish users voluntarily participate in any action determination mechanism that aims to optimize the social welfare? In a nutshell, the prior work of [3], [4] analyzed specific games, with linear cost functions, for local public good provision, whereas our work focusses on *designing a mechanism that can induce, via nonlinear tax functions, "appropriate" games among the network users so as to implement the optimum social welfare in NE*. It is this difference in the tax functions that distinguishes our results from those of [3], [4].

Previous works on implementation approach (Nash implementation) for (pure) public goods can be found in [9], [10], [11], [12]. For our work, we obtained inspiration from [10]. In [10] Hurwicz presents a Nash implementation mechanism that implements the Lindahl allocation (optimum social welfare) for a public good economy. Hurwicz' mechanism also possesses the properties of individual rationality (i.e. it induces the selfish users to voluntarily participate in the mechanism) and budget balance (i.e. it balances the flow of money in the system). A local public good network can be thought of as a limiting case of a public good network, in which the influence of each public good tends to vanish on a subset of network users. However, taking the corresponding limits in the Hurwicz' mechanism does not result in a local public good provisioning mechanism with all the original properties of the Hurwicz' mechanism. In particular, such a limiting mechanism does not retain the budget balance property which is very important to avoid any scarcity/wastage of money. In this paper we address the problem of designing a local public good provisioning mechanism that possesses the desirable properties of budget balance, individual rationality, and Nash implementation of optimum social welfare. The mechanism we develop is more general than Hurwicz' mechanism; Hurwicz' mechanism can be obtained as a special case of our mechanism by setting $\mathcal{R}_i = \mathcal{C}_j = \mathcal{N}\ \forall\ i, j, \in \mathcal{N}$ (the special case where all users' actions affect all users' utilities) in our mechanism. Our mechanism also proivdes a more efficient way to achieve the properties of Nash implementation, individual rationality, and budget balance as it uses, in general, a much smaller message space than Hurwicz' mechanism. To the best of our knowledge the resource allocation problem and its solution that we present in this paper is the first attempt to analyze a local public goods network model in the framework of implementation theory. Below we state our contributions.

B. Contribution of the paper

The key contributions of this paper are: 1) The formulation of a problem of local public goods provisioning in the framework of implementation theory. 2) The specification of a game form [2] (decentralized mechanism) for the above problem that, (i) implements in NE the optimal solution of the corresponding centralized local public good provisioning problem; (ii) is individually rational; [3] and (iii) results in budget balance at all NE and off equilibrium.

The rest of the paper is organized as follows. In Section II-A we present the model of local public good network. In Section II-B we formulate the local public good provisioning problem. In Section III-A we present a game form for this problem and discuss its properties in Section III-B. We conclude in Section IV with a discussion on future directions. **Notation used in the paper:** We use bold font to represent vectors and normal font for scalars. We use bold uppercase letters to represent matrices. We represent the element of a vector by a subscript on the vector symbol, and the element of a matrix by double subscript on the matrix symbol. To denote the vector whose elements are all x_i such that $i \in \mathcal{S}$ for some set $\mathcal{S}$, we use the notation $(x_i)_{i\in\mathcal{S}}$ and we abbreviate it as $\boldsymbol{x}_{\mathcal{S}}$. We treat bold $\mathbf{0}$ as a zero vector of appropriate size which is determined by the context. We use the notation $(x_i, \boldsymbol{x}^*/i)$ to represent a vector of dimension same as that of $\boldsymbol{x}^*$, whose ith element is x_i and all other elements are the same as the corresponding elements of $\boldsymbol{x}^*$. We represent a diagonal matrix of size $N \times N$ whose diagonal entries are elements of the vector $\boldsymbol{x} \in \mathbb{R}^N$ by $\text{diag}(\boldsymbol{x})$.

II. The local public good provisioning problem

In this section we present a model of local public good network motivated by various applications such as wireless communication, online advertising [13], social and information networks [4], [3]. We first describe the components of the model and the assumptions we make on the properties of the network. We then present a resource allocation problem for this model and formulate it as an optimization problem.

A. The network model (M)

We consider a network consisting of N users and one network operator. Let the set of users be $\mathcal{N} := \{1, 2, \ldots, N\}$. Each user $i \in \mathcal{N}$ has to take an action $a_i \in \mathcal{A}_i$ where $\mathcal{A}_i$ is the set that specifies user i's feasible actions. In a real network, a user's actions can be consumption/generation of resources or decisions regarding various tasks. We assume that,

[1]Refer to [5], [6], [7] and [8, Chapter 3] for an introduction to implementation theory.

[2]See [8, Chapter 3] and [7], [6], [5] for the definition of "game form".

[3]Refer to [8, Chapter 3] and [7] for the definition of "individual rationality" and "implementation in NE."

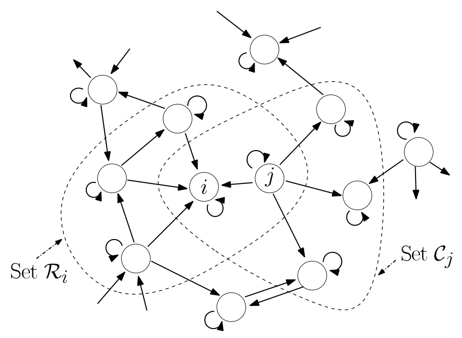

Fig. 1. A local public good network depicting the Neighbor sets $\mathcal{R}_i$ and $\mathcal{C}_j$ of users i and j respectively.

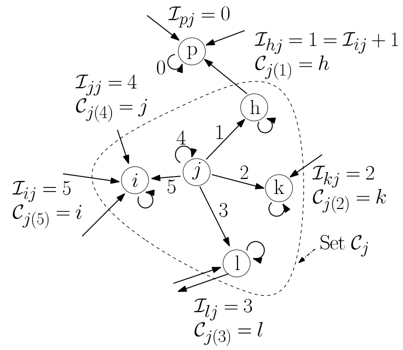

Fig. 2. Illustration of indexing rule for set $\mathcal{C}_j$ shown in Fig. 1. Index $\mathcal{I}_{rj}$ of user $r \in \mathcal{C}_j$ is indicated on the arrow directed from j to r. The notation to denote these indices and to denote the user with a particular index is shown outside the dashed boundary demarcating the set $\mathcal{C}_j$.

Assumption 1: For all $i \in \mathcal{N}$, $\mathcal{A}_i$ is a convex and compact set in $\mathbb{R}$ that includes 0. [4] Furthermore, for each user $i \in \mathcal{N}$, the set $\mathcal{A}_i$ is its private information, i.e. $\mathcal{A}_i$ is known only to user i and nobody else in the network.

Because of the users' interactions in the network, the actions taken by a user directly affect the performance of other users in the network. Thus, the performance of the network is determined by the collective actions of all users. We assume that the network is large-scale, therefore, every user's actions directly affect only a subset of network users in $\mathcal{N}$. Thus we can treat each user's action as a local public good. We depict the above feature by a directed graph as shown in Fig. 1. In the graph, an arrow from j to i indicates that user j affects user i; we represent the same in the text as $j \to i$. We assume that $i \to i$ for all $i \in \mathcal{N}$.

Mathematically, we denote the set of users that affect user i by $\mathcal{R}_i := \{k \in \mathcal{N} \mid k \to i\}$. Similarly, we denote the set of users that are affected by user j by $\mathcal{C}_j := \{k \in \mathcal{N} \mid j \to k\}$. We represent the interactions of all network users together by a graph matrix $\boldsymbol{G} := [G_{ij}]_{N \times N}$. The matrix $\boldsymbol{G}$ consists of 0's and 1's, where $G_{ij} = 1$ represents that user i is affected by user j, i.e. $j \in \mathcal{R}_i$ and $G_{ij} = 0$ represents no influence of user j on user i, i.e. $j \notin \mathcal{R}_i$. Note that $\boldsymbol{G}$ need not be a symmetric matrix. Because $i \to i$, $G_{ii} = 1$ for all $i \in \mathcal{N}$. We assume that,

Assumption 2: The sets $\mathcal{R}_i, \mathcal{C}_i, i \in \mathcal{N}$, are independent of the users' action profile $\boldsymbol{a}_{\mathcal{N}} := (a_k)_{k \in \mathcal{N}} \in \prod_{k \in \mathcal{N}} \mathcal{A}_k$. Furthermore, for each $i \in \mathcal{N}$, $|\mathcal{C}_i| \geq 3$.

We consider the condition $|\mathcal{C}_i| \geq 3, i \in \mathcal{N}$, so as to ensure construction of a mechanism that is budget balanced at all possible allocations, those that correspond to Nash equilibria as well as those that correspond to off-equilibrium messages. For examples of applications where Assumption 2 holds, see [13], [4], [3].

We assume that,

Assumption 3: Each user $i \in \mathcal{N}$ knows that the set of feasible actions $\mathcal{A}_j$ of each of its neighbors $j \in \mathcal{R}_i$ is a convex and compact subset of $\mathbb{R}$ that includes 0.

The performance of a user that results from actions taken by the users affecting it is quantified by a utility function. We denote the utility of user $i \in \mathcal{N}$ resulting from the action profile $\boldsymbol{a}_{\mathcal{R}_i} := (a_k)_{k \in \mathcal{R}_i}$ by $u_i(\boldsymbol{a}_{\mathcal{R}_i})$. We assume that,

Assumption 4: For all $i \in \mathcal{N}$, the utility function $u_i : \mathbb{R}^{|\mathcal{R}_i|} \to \mathbb{R} \cup \{-\infty\}$ is concave in $\boldsymbol{a}_{\mathcal{R}_i}$ and $u_i(\boldsymbol{a}_{\mathcal{R}_i}) = -\infty$ for $a_i \notin \mathcal{A}_i$. [5] The function u_i is user i's private information. The assumptions that u_i is concave and is user i's private information are motivated by applications described in [13], [4], [3]. The assumption that $u_i(\boldsymbol{a}_{\mathcal{R}_i}) = -\infty$ for $a_i \notin \mathcal{A}_i$ captures the fact that an action profile $(\boldsymbol{a}_{\mathcal{R}_i})$ is of no significance to user i if $a_i \notin \mathcal{A}_i$. We assume that,

Assumption 5: Each network user $i \in \mathcal{N}$ is selfish, non-cooperative, and strategic.

Assumption 5 implies that the users have an incentive to misrepresent their private information, e.g. a user $i \in \mathcal{N}$ may not want to report to other users or to the network operator its true preference for the users' actions, if this results in an action profile in its own favor.

Each user $i \in \mathcal{N}$ pays a tax $t_i \in \mathbb{R}$ to the network operator. This tax can be imposed for the following reasons: (i) For the use of the network by the users. (ii) To provide incentives to the users to take actions that achieve a network-wide performance objective. The tax is set according to the rules specified by a mechanism and it can be either positive or negative for a user. With the flexibility of either charging a user (positive tax) or paying compensation/subsidy (negative tax) to a user, it is possible to induce the users to behave in a way such that a network-wide performance objective is achieved. For example, in a network with limited resources, we can set "positive tax" for the users that receive resources close to the amounts requested by them and we can pay "compensation" to the users that receive resources that are not close to their desirable ones. Thus, with the available resources, we can satisfy all the users and induce them to behave in a way that leads to a resource allocation that is optimal according to a network-wide performance criterion. We assume that,

Assumption 6: The network operator does not have any utility associated with the users' actions or taxes. It does not derive any profit from the users' taxes and acts like an accountant that redistributes the tax among the users according to the specifications of the allocation mechanism.

[4]In this paper we assume the sets $\mathcal{A}_i, i \in \mathcal{N}$, to be in $\mathbb{R}$ for simplicity. However, the decentralized mechanism and the results we present in this paper can be easily generalized to the scenario where for each $i \in \mathcal{N}$, $\mathcal{A}_i \subset \mathbb{R}^{n_i}$ is a convex and compact set in higher dimensional space $\mathbb{R}^{n_i}$. Furthermore, each space $\mathbb{R}^{n_i}$ can be of a different dimension n_i for different $i \in \mathcal{N}$.

[5]Note that a_i is always an element of $\boldsymbol{a}_{\mathcal{R}_i}$ because $i \to i$ and hence $i \in \mathcal{R}_i$.

Assumption 6 implies that tax is charged in a way such that

$$\sum_{i\in\mathcal{N}} t_i = 0. \quad (1)$$

To describe the "overall satisfaction" of a user from the performance it receives from all users' actions and the tax it pays for it, we define an "aggregate utility function" $u_i^A(\boldsymbol{a}_{\mathcal{R}_i}, t_i) : \mathbb{R}^{|\mathcal{R}_i|+1} \to \mathbb{R} \cup \{-\infty\}$ for each user $i \in \mathcal{N}$:

$$u_i^A(\boldsymbol{a}_{\mathcal{R}_i}, t_i) := \begin{cases} -t_i + u_i(\boldsymbol{a}_{\mathcal{R}_i}), & \text{if } a_i \in \mathcal{A}_i, a_j \in \mathbb{R}, j \in \mathcal{R}_i\backslash\{i\}, \\ -\infty, & \text{otherwise.} \end{cases} \quad (2)$$

Because u_i and $\mathcal{A}_i$ are user i's private information (Assumptions 1 and 4), the aggregate utility u_i^A is also user i's private information. As stated in Assumption 5, users are non-cooperative and selfish. Therefore, *the users are self aggregate utility maximizers*.

In this paper we restrict attention to static problems, i.e.,

Assumption 7: The set $\mathcal{N}$ of users, the graph $\boldsymbol{G}$, users' action spaces $\mathcal{A}_i, i \in \mathcal{N}$, and their utility functions $u_i, i \in \mathcal{N}$, are fixed in advance and they do not change during the time period of interest.

We also assume that,

Assumption 8: Every user $i \in \mathcal{N}$ knows the set $\mathcal{R}_i$ of users that affect it as well as the set $\mathcal{C}_i$ of users that are affected by it. The network operator knows $\mathcal{R}_i$ and $\mathcal{C}_i$ for all $i \in \mathcal{N}$.

In networks where the sets $\mathcal{R}_i$ and $\mathcal{C}_i$ are not known to the users beforehand, Assumption 8 is still reasonable because of the following reason. As the graph $\boldsymbol{G}$ does not change during the time period of interest (Assumption 7), the information about the neighbor sets $\mathcal{R}_i$ and $\mathcal{C}_i, i \in \mathcal{N}$, can be passed to the respective users by the network operator before the users determine their actions. Alternatively, the users can themselves determine the set of their neighbors before determining their actions. [6] Thus, Assumption 8 can hold true for the rest of the action determination process. In the next section we present a local public good provisioning problem for Model (M).

B. Decentralized local public good provision problem (P_D)

For the network model (M) we wish to develop a mechanism to determine the users' action and tax profiles $(\boldsymbol{a}_{\mathcal{N}}, \boldsymbol{t}_{\mathcal{N}}) := ((a_1, a_2, \ldots, a_N), (t_1, t_2, \ldots, t_N))$. We want the mechanism to work under the decentralized information constraints of the model and to lead to a solution to the following centralized problem.

The centralized problem (P_C)

$$\begin{aligned} &\max_{(\boldsymbol{a}_{\mathcal{N}}, \boldsymbol{t}_{\mathcal{N}})} \quad \sum_{i\in\mathcal{N}} u_i^A(\boldsymbol{a}_{\mathcal{R}_i}, t_i) \\ &\quad\text{s.t.} \quad \sum_{i\in\mathcal{N}} t_i = 0 \end{aligned} \quad (3)$$

$$\begin{aligned} \equiv &\max_{(\boldsymbol{a}_{\mathcal{N}}, \boldsymbol{t}_{\mathcal{N}})\in\mathcal{D}} \sum_{i\in\mathcal{N}} u_i(\boldsymbol{a}_{\mathcal{R}_i}), \quad \text{where} \\ &\mathcal{D} := \{(\boldsymbol{a}_{\mathcal{N}}, \boldsymbol{t}_{\mathcal{N}}) \in \mathbb{R}^{2N} \mid a_i \in \mathcal{A}_i \ \forall\, i \in \mathcal{N}; \sum_{i\in\mathcal{N}} t_i = 0\} \end{aligned} \quad (4)$$

[6] The exact method by which the users get information about their neighbor sets in a real network depends on the network characteristics.

The centralized optimization problem (3) is equivalent to (4) because for $(\boldsymbol{a}_{\mathcal{N}}, \boldsymbol{t}_{\mathcal{N}}) \notin \mathcal{D}$, the objective function in (3) is negative infinity by (2). Thus $\mathcal{D}$ is the set of feasible solutions of Problem (P_C). Since by Assumption 4, the objective function in (4) is concave in $\boldsymbol{a}_{\mathcal{N}}$ and the sets $\mathcal{A}_i, i \in \mathcal{N}$, are convex and compact, there exists an optimal action profile $\boldsymbol{a}_{\mathcal{N}}^*$ for (P_C). Furthermore, since the objective function in (4) does not explicitly depend on $\boldsymbol{t}_{\mathcal{N}}$, an optimal solution of (P_C) must be of the form $(\boldsymbol{a}_{\mathcal{N}}^*, \boldsymbol{t}_{\mathcal{N}})$, where $\boldsymbol{t}_{\mathcal{N}}$ is any feasible tax profile for (P_C), i.e. a tax profile that satisfies (1).

The solutions of Problem (P_C) are ideal action and tax profiles that we would like to obtain. If there exists an entity that has centralized information about the network, i.e. it knows all the utility functions $u_i, i \in \mathcal{N}$, and all action spaces $\mathcal{A}_i, i \in \mathcal{N}$, then that entity can compute the above ideal profiles by solving Problem (P_C). Therefore, we call the solutions of Problem (P_C) optimal centralized allocations. In the network described by Model (M), there is no entity that knows perfectly all the parameters that describe Problem (P_C) (Assumptions 1 and 4). Therefore, we need to develop a mechanism that allows the network users to communicate with one another and that leads to optimal solutions of Problem (P_C). Since a key assumption in Model (M) is that the users are strategic and non-cooperative, the mechanism we develop must take into account the users' strategic behavior in their communication with one another. To address all of these issues we take the approach of implementation theory [5] for the solution of the decentralized local public good provisioning problem for Model (M). Henceforth we call this decentralized allocation problem as Problem (P_D). In the next section we present a decentralized mechanism (game form) for local public good provisioning that works under the constraints imposed by Model (M) and achieves optimal centralized allocations.

III. A DECENTRALIZED LOCAL PUBLIC GOOD PROVISIONING MECHANISM

For Problem (P_D), we want to develop a game form (message space and outcome function) that is *individually rational*, *budget balanced*, and that *implements in Nash equilibria* the goal correspondence defined by the solution of Problem (P_C). [7] Individual rationality guarantees voluntary participation of the users in the allocation process specified by the game form, budget balance guarantees that there is no money left unclaimed/unallocated at the end of the allocation process (i.e. it ensures (1)), and implementation in NE guarantees that the allocations corresponding to the set of NE of the game induced by the game form are a subset of the optimal centralized allocations (solutions of Problem (P_C)).

We would like to clarify at this point the definition of individual rationality (voluntary participation) in the context of our problem. Note that in the network model (M), the participation/non-participation of each user determines the

[7] The definition of game form, goal correspondence, individual rationality, budget balance and implementation in Nash equilibria is given in [8, Chapter 3].

network structure and the set of local public goods (users' actions) affecting the participating users. To define individual rationality in this setting we consider our mechanism to be consisting of two stages as discussed in [14, Chapter 7]. In the first stage, knowing the game form, each user makes a decision whether to participate in the game form or not. The users who decide not to participate are considered out of the system. Those who decide to participate follow the game form to determine the levels of local public goods in the network formed by them. [8] In such a two stage mechanism, individual rationality implies the following. If the network formed by the participating users satisfies all the properties of Model (M), [9] then, at all NE of the game induced by the game form among the participating users, the utility of each participating user will be at least as much as its utility without participation (i.e. if it is out of the system).

We would also like to clarify the rationale behind choosing NE as the solution concept for our problem. Note that because of assumptions 1 and 4 in Model (M), the environment of our problem is one of incomplete information. Therefore one may speculate the use of Bayesian Nash or dominant strategy as appropriate solution concepts for our problem. However, since the users in Model (M) do not possess any prior beliefs about the utility functions and action sets of other users, we cannot use Bayesian Nash as a solution concept for Model (M). Furthermore, because of impossibility results for the existence of non-parametric efficient dominant strategy mechanisms in classical public good environments [15], we do not know if it is possible to design such mechanisms for the local public good environment of Model (M). The well known Vickrey-Clarke-Groves (VCG) mechanisms that achieve incentive compatibility and efficiency with respect to non-numeraire goods, do not guarantee budget balance [15]. Hence they are inappropriate for our problem as budget balance is one of the desirable properties in our problem. VCG mechanisms are also unsuitable for our problem because they are direct mechanisms and any direct mechanism would require infinite message space to communicate the generic continuous (and concave) utility functions of users in Model (M). Because of all of above reasons, and the known existence results for non-parametric, individually rational, budget-balanced Nash implementation mechanisms for classical private and public goods environments [15], we choose Nash as the solution concept for our problem. We adopt Nash's original "mass action" interpretation of NE [16, page 21]. Implicit in this interpretation is the assumption that the problems's environment is stable, that is, it does not change before the agents reach their equilibrium strategies. This assumption is consistent with our Assumption 7. Nash's "mass action" interpretation of NE has also been adopted in [15, pp. 69-70], [17, page 664], [7], and [18], [19]. Specifically, by quoting [17], "we interpret our analysis as applying to an unspecified (message exchange) process in which users grope their way to a stationary message and in which the Nash property is a necessary condition for stationarity."

We next construct a game form for the resource allocation problem (P_D) that achieves the abovementioned desirable properties – Nash implementation, individual rationality, and budget balance.

A. The game form

In this section we present a game form for the local public good provisioning problem presented in Section II-B. We provide explicit expressions of each of the components of the game form, the message space and the outcome function. We assume that the game form is common knowledge among the users and the network operator.

The message space: Each user $i \in \mathcal{N}$ sends to the network operator a message $\boldsymbol{m}_i \in \mathbb{R}^{|\mathcal{R}_i|} \times \mathbb{R}_+^{|\mathcal{R}_i|} =: \mathcal{M}_i$ of the following form:

$$\boldsymbol{m}_i := ({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}); \quad {}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|},\ {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}, \tag{5}$$

$$\text{where, } {}^i\boldsymbol{a}_{\mathcal{R}_i} := ({}^ia_k)_{k\in\mathcal{R}_i};\ {}^i\boldsymbol{\pi}_{\mathcal{R}_i} := ({}^i\pi_k)_{k\in\mathcal{R}_i},\ i \in \mathcal{N}. \tag{6}$$

User i also sends the component $({}^ia_k, {}^i\pi_k), k \in \mathcal{R}_i$, of its message to its neighbor $k \in \mathcal{R}_i$. In this message, ${}^ia_k$ is the action proposal for user $k, k \in \mathcal{R}_i$, by user $i, i \in \mathcal{N}$. Similarly, ${}^i\pi_k$ is the price that user $i, i \in \mathcal{N}$, proposes to pay for the action of user $k, k \in \mathcal{R}_i$. A detailed interpretation of these message elements is given in Section III-B.

The outcome function: After the users communicate their messages to the network operator, their actions and taxes are determined as follows. For each user $i \in \mathcal{N}$, the network operator determines the action $\hat{a}_i$ of user i from the messages communicated by its neighbors that are affected by it (set $\mathcal{C}_i$), i.e. from the message profile $\boldsymbol{m}_{\mathcal{C}_i} := (\boldsymbol{m}_k)_{k\in\mathcal{C}_i}$:

$$\hat{a}_i(\boldsymbol{m}_{\mathcal{C}_i}) = \frac{1}{|\mathcal{C}_i|} \sum_{k\in\mathcal{C}_i} {}^ka_i, \quad i \in \mathcal{N}. \tag{7}$$

To determine the users' taxes the network operator considers each set $\mathcal{C}_j$, $j \in \mathcal{N}$, and assigns indices $1, 2, \ldots, |\mathcal{C}_j|$ in a cyclic order to the users in $\mathcal{C}_j$. Each index $1, 2, \ldots, |\mathcal{C}_j|$ is assigned to an arbitrary but unique user $i \in \mathcal{C}_j$. Once the indices are assigned to the users in each set $\mathcal{C}_j$, they remain fixed throughout the time period of interest. We denote the index of user i associated with set $\mathcal{C}_j$ by $\mathcal{I}_{ij}$. The index $\mathcal{I}_{ij} \in \{1, 2, \ldots, |\mathcal{C}_j|\}$ if $i \in \mathcal{C}_j$, and $\mathcal{I}_{ij} = 0$ if $i \notin \mathcal{C}_j$. Since for each set $\mathcal{C}_j$, each index $1, 2, \ldots, |\mathcal{C}_j|$ is assigned to a unique user $i \in \mathcal{C}_j$, therefore, $\forall\ i, k \in \mathcal{C}_j$ such that $i \neq k$, $\mathcal{I}_{ij} \neq \mathcal{I}_{kj}$. Note also that for any user $i \in \mathcal{N}$, and any $j, k \in \mathcal{R}_i$, the indices $\mathcal{I}_{ij}$ and $\mathcal{I}_{ik}$ are not necessarily the same and are independent of each other. We denote the user with index $k \in \{1, 2, \ldots, |\mathcal{C}_j|\}$ in set $\mathcal{C}_j$ by $\mathcal{C}_{j(k)}$. Thus, $\mathcal{C}_{j(\mathcal{I}_{ij})} = i$ for $i \in \mathcal{C}_j$. The cyclic order indexing means that, if $\mathcal{I}_{ij} = |\mathcal{C}_j|$, then $\mathcal{C}_{j(\mathcal{I}_{ij}+1)} = \mathcal{C}_{j(1)}$, $\mathcal{C}_{j(\mathcal{I}_{ij}+2)} = \mathcal{C}_{j(2)}$, and so on. In Fig. 2 we illustrate the above indexing rule for the set $\mathcal{C}_j$ shown in Fig. 1.

[8]This network is a subgraph obtained by removing the nodes corresponding to non-participating users from the original graph (directed network) constructed by all the users in the system.

[9]In particular, the network formed by the participating users must satisfy Assumption 2 that there are at least three users affected by each local public good in this network. Note that all other assumptions of Model (M) automatically carry over to the network formed by any subset of the users in Model (M).

Based on the above indexing, the users' taxes $\hat{t}_i, i \in \mathcal{N}$, are determined as follows.

$$\hat{t}_i((\boldsymbol{m}_{\mathcal{C}_j})_{j\in\mathcal{R}_i})=\sum_{j\in\mathcal{R}_i} l_{ij}(\boldsymbol{m}_{\mathcal{C}_j})\ \hat{a}_j(\boldsymbol{m}_{\mathcal{C}_j})+\sum_{j\in\mathcal{R}_i} {}^{i}\pi_j\left({}^{i}a_j- {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j\right)^2-\sum_{j\in\mathcal{R}_i} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j\left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j\right)^2 \quad (8)$$

$$\text{where, } l_{ij}(\boldsymbol{m}_{\mathcal{C}_j})= {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j- {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}\pi_j,\ j\in\mathcal{R}_i,\ i\in\mathcal{N}. \quad (9)$$

We would like to emphasize here that the presence of the network operator is necessary for strategy-proofness and implementation of the above game form. A detailed discussion on the need and significance of the network operator can be found in [13].

The game form given by (5)–(9) and the users' aggregate utility functions in (2) induce a game $(\times_{i\in\mathcal{N}}\mathcal{M}_i, (\hat{a}_i, \hat{t}_i)_{i\in\mathcal{N}}, \{u_i^A\}_{i\in\mathcal{N}})$. In this game, the set of network users $\mathcal{N}$ are the players, the set of strategies of a user is its message space $\mathcal{M}_i$, and a user's payoff is its utility $u_i^A\Big(\big(\hat{a}_j(\boldsymbol{m}_{\mathcal{C}_j})\big)_{j\in\mathcal{R}_i},\ \hat{t}_i\big((\boldsymbol{m}_{\mathcal{C}_j})_{j\in\mathcal{R}_i}\big)\Big)$ that it obtains at the allocation determined by the communicated messages. We define a NE of this game as a message profile $\boldsymbol{m}_{\mathcal{N}}^*$ that has the following property: $\forall\ i \in \mathcal{N}$ and $\forall\ \boldsymbol{m}_i \in \mathcal{M}_i$,

$$\begin{aligned} &u_i^A\Big(\big(\hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})\big)_{j\in\mathcal{R}_i},\ \hat{t}_i\big((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}\big)\Big) \geq \\ &u_i^A\Big(\big(\hat{a}_j(\boldsymbol{m}_i, \boldsymbol{m}^*_{\mathcal{C}_j}/i)\big)_{j\in\mathcal{R}_i},\ \hat{t}_i\big((\boldsymbol{m}_i, \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big)\Big). \end{aligned} \quad (10)$$

As discussed earlier, NE in general describe strategic behavior of users in games of complete information. This can be seen from (10) where, to define a NE, it requires complete information of all users' aggregate utility functions. However, the users in Model (M) do not know each other's utilities; therefore, the game induced by the game form (5)–(9) and the users' aggregate utility functions (2) is not one of complete information. Therefore, for our problem we adopt the NE interpretation of [17] and [15, Section 4] as discussed at the beginning of Section III. That is, we interpret NE as the "stationary" messages of an unspecified (message exchange) process that are characterized by the Nash property (10).

In the next section we show that the allocations obtained by the game form presented in (5)–(9) at all NE message profiles (satisfying (10)), are optimal centralized allocations.

B. Properties of the game form

We begin this section with an intuitive discussion on how the game form presented in Section III-A achieves optimal centralized allocations. We then formalize the results in Theorems 1 and 2.

To understand how the proposed game form achieves optimal centralized allocations, let us look at the properties of NE allocations corresponding to this game form. A NE of the game induced by the game form (5)–(9) and the users' utility functions (2) can be interpreted as follows: Given the users' messages $\boldsymbol{m}_k, k \in \mathcal{C}_i$, the outcome function computes user i's action as $1/|\mathcal{C}_i|\big(\sum_{k\in\mathcal{C}_i} {}^{k}a_i\big)$. Therefore, user i's action proposal ${}^{i}a_i$ can be interpreted as the increment that i desires over the sum of other users' action proposals for i, so as to bring its allocated action $\hat{a}_i$ to its own desired value. Thus, if the action computed for i based on its neighbors' proposals does not lie in $\mathcal{A}_i$, user i can propose an appropriate action ${}^{i}a_i$ and bring its allocated action within $\mathcal{A}_i$. The flexibility of proposing any action ${}^{i}a_i \in \mathbb{R}$ gives each user $i \in \mathcal{N}$ the capability to bring its allocation $\hat{a}_i$ within its feasible set $\mathcal{A}_i$ by unilateral deviation. Therefore, at any NE, $\hat{a}_i \in \mathcal{A}_i,\ \forall\ i \in \mathcal{N}$. By taking the sum of taxes in (8) it can further be seen, after some computations, that the allocated tax profile $(\hat{t}_i)_{i\in\mathcal{N}}$ satisfies (1) (even at off-NE messages). Thus, all NE allocations $\Big((\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}))_{i\in\mathcal{N}},\ (\hat{t}_i((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}))_{i\in\mathcal{N}}\Big)$ lie in $\mathcal{D}$ and hence are feasible solutions of Problem (P_C).

To see further properties of NE allocations, let us look at the tax function in (8). The tax of user i consists of three types of terms. The type-1 term is $\sum_{j\in\mathcal{R}_i} l_{ij}(\boldsymbol{m}_{\mathcal{C}_j})\ \hat{a}_j(\boldsymbol{m}_{\mathcal{C}_j})$; it depends on all action proposals for each of user i's neighbors $j \in \mathcal{R}_i$, and the price proposals for each of these neighbors by users other than user i. The type-2 term is $\sum_{j\in\mathcal{R}_i} {}^{i}\pi_j\left({}^{i}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j\right)^2$; this term depends on ${}^{i}\boldsymbol{a}_{\mathcal{R}_i}$ as well as ${}^{i}\boldsymbol{\pi}_{\mathcal{R}_i}$. Finally, the type-3 term is the following: $-\sum_{j\in\mathcal{R}_i} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j\times\left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j\right)^2$; this term depends only on the messages of users other than i. Since ${}^{i}\boldsymbol{\pi}_{\mathcal{R}_i}$ does not affect the determination of user i's action, and affects only the type-2 term in $\hat{t}_i$, the NE strategy of user $i, i \in \mathcal{N}$, that minimizes its tax is to propose for each $j \in \mathcal{R}_i$, ${}^{i}\boldsymbol{\pi}_j = 0$ unless at the NE, ${}^{i}a_j = {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j$. Since the type-2 and type-3 terms in the users' tax are similar across users, for each $i \in \mathcal{N}$ and $j \in \mathcal{R}_i$, all the users $k \in \mathcal{C}_j$ choose the above strategy at NE. Therefore, the type-2 and type-3 terms vanish from every users' tax $\hat{t}_i, i \in \mathcal{N}$, at all NE. Thus, the tax that each user $i \in \mathcal{N}$ pays at a NE $\boldsymbol{m}^*_{\mathcal{N}}$ is of the form $\sum_{j\in\mathcal{R}_i} l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j})\ \hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})$. The NE term $l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j}), i \in \mathcal{N}, j \in \mathcal{R}_i$, can therefore be interpreted as the "personalized price" for user i for the NE action $\hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})$ of its neighbor j. Note that at a NE, the personalized price for user i is not controlled by i's own message. The reduction of the users' NE taxes into the form $\sum_{j\in\mathcal{R}_i} l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j})\ \hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})$ implies that at a NE, each user $i \in \mathcal{N}$ has a control over its aggregate utility only through its action proposal. [10] If all other users' messages are fixed, each user has the capability of shifting the allocated action profile $\hat{\boldsymbol{a}}_{\mathcal{R}_i}$ to its desired value by proposing an appropriate ${}^{i}\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}$ (See the discussion in the previous paragraph). Therefore, the NE strategy of each user $i \in \mathcal{N}$ is to propose an action profile ${}^{i}\boldsymbol{a}_{\mathcal{R}_i}$ that results in an allocation $\hat{\boldsymbol{a}}_{\mathcal{R}_i}$ that maximizes its aggregate utility. Thus, at a NE, each user maximizes its aggregate utility for its given personalized prices. By the construction of the tax function, the sum of the users' tax is zero at all NE and off equilibria. Thus, the individual aggregate utility maximization of the users also result in the maximization of the sum of users' aggregate utilities subject to the budget balance constraint which is the objective of Problem (P_C).

It is worth mentioning at this point the significance of type-2 and type-3 terms in the users' tax. As explained above, these

[10] Note that user i's action proposal determines the actions of all the users $j \in \mathcal{R}_i$; thus, it affects user i's utility $u_i\Big(\big(\hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})\big)_{j\in\mathcal{R}_i}\Big)$ as well as its tax $\sum_{j\in\mathcal{R}_i} l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j})\ \hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})$.

two terms vanish at NE. However, if for some user $i \in \mathcal{N}$ these terms are not present in its tax $\hat{t}_i$, then, the price proposal ${}^i\boldsymbol{\pi}_{\mathcal{R}_i}$ of user i will not affect its tax and hence, its aggregate utility. In such a case, user i can propose arbitrary prices ${}^i\boldsymbol{\pi}_{\mathcal{R}_i}$ because they would affect only other users' NE prices. The presence of type-2 and type-3 terms in user i's tax prevent such a behavior as they impose a penalty on user i if it proposes a high value of ${}^i\boldsymbol{\pi}_{\mathcal{R}_i}$ or if its action proposal for its neighbors deviates too much from other users' proposals. Even though the presence of type-2 and type-3 terms in user i's tax is necessary as explained above, it is important that the NE price $l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j}), j \in \mathcal{R}_i$ of user $i \in \mathcal{N}$ is not affected by i's own proposal ${}^i\boldsymbol{\pi}_{\mathcal{R}_i}$. This is because, in such a case, user i may influence its own NE price in an unfair manner and may not behave as a price taker. To avoid such a situation, the type-2 and type-3 terms are designed in a way so that they vanish at NE. Thus, this construction induces price taking behavior in the users at NE and leads to optimal allocations.

The results that formally establish the above properties of the game form are summarized in Theorems 1 and 2 below.

Theorem 1: Let $\boldsymbol{m}^*_{\mathcal{N}}$ be a NE of the game induced by the game form presented in Section III-A and the users' utility functions (2). Let $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}}) := (\hat{\boldsymbol{a}}_{\mathcal{N}}(\boldsymbol{m}^*_{\mathcal{N}}), \hat{\boldsymbol{t}}_{\mathcal{N}}(\boldsymbol{m}^*_{\mathcal{N}})) := \Big((\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}))_{i\in\mathcal{N}}, (\hat{t}_i((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}))_{i\in\mathcal{N}}\Big)$ be the action and tax profiles at $\boldsymbol{m}^*_{\mathcal{N}}$ determined by the game form. Then,

(a) Each user $i \in \mathcal{N}$ weakly prefers its allocation $(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i)$ to the initial allocation $(\boldsymbol{0}, 0)$. Mathematically,

$$u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i\Big) \geq u_i^A\Big(\boldsymbol{0},\, 0\Big), \qquad \forall\, i \in \mathcal{N}.$$

(b) $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$ is an optimal solution of Problem (P_C). ⊠

Theorem 2: Let $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ be an optimum action profile corresponding to Problem (P_C). Then,

(a) There exist a set of personalized prices $l^*_{ij}, j \in \mathcal{R}_i, i \in \mathcal{N}$, such that

$$\hat{\boldsymbol{a}}^*_{\mathcal{R}_i} = \underset{\substack{\hat{a}_i \in \mathcal{A}_i \\ \hat{a}_j \in \mathbb{R},\, j \in \mathcal{R}_i\setminus\{i\}}}{\arg\max} \; -\sum_{j\in\mathcal{R}_i} l^*_{ij}\, \hat{a}_j + u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}), \qquad \forall\, i \in \mathcal{N}.$$

(b) There exists at least one NE $\boldsymbol{m}^*_{\mathcal{N}}$ of the game induced by the game form presented in Section III-A and the users' utility functions (2) such that, $\hat{\boldsymbol{a}}_{\mathcal{N}}(\boldsymbol{m}^*_{\mathcal{N}}) = \hat{\boldsymbol{a}}^*_{\mathcal{N}}$. Furthermore, if $\hat{t}^*_i := \sum_{j\in\mathcal{R}_i} l^*_{ij}\hat{a}^*_j, i \in \mathcal{N}$, the set of all NE $\boldsymbol{m}^*_{\mathcal{N}} = (\boldsymbol{m}^*_i)_{i\in\mathcal{N}} = ({}^i\boldsymbol{a}^*_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}^*_{\mathcal{R}_i})$ that result in $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$ is characterized by the solution of the following set of conditions:

$$\begin{aligned} \frac{1}{|\mathcal{C}_i|}\sum_{k\in\mathcal{C}_i} {}^k a^*_i &= \hat{a}^*_i, \quad i \in \mathcal{N}, \\ {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}\pi^*_j &= l^*_{ij}, \quad j \in \mathcal{R}_i,\ i \in \mathcal{N}, \\ {}^i\pi^*_j \left({}^i a^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}} a^*_j\right)^2 &= 0, \quad j \in \mathcal{R}_i, \quad i \in \mathcal{N}, \\ {}^i\pi^*_j &\geq 0, \quad j \in \mathcal{R}_i, \quad i \in \mathcal{N}. \boxtimes \end{aligned}$$

Because Theorem 1 is stated for an arbitrary NE $\boldsymbol{m}^*_{\mathcal{N}}$ of the game induced by the game form of Section III-A and the users' utility functions (2), the assertion of the theorem holds for all NE of this game. Thus, part (a) of Theorem 1 establishes that the game form presented in Section III-A is *individually rational*, i.e., at any NE allocation, the aggregate utility of each user is at least as much as its aggregate utility before participating in the game/allocation process. Because of this property of the game form, each user voluntarily participates in the allocation process. Part (b) of Theorem 1 asserts that all NE of the game induced by the game form of Section III-A and the users' utility functions (2) result in optimal centralized allocations (solutions of Problem (P_C)). Thus the set of NE allocations is a subset of the set of optimal centralized allocations. This establishes that the game form of Section III-A *implements in NE* the goal correspondence defined by the solutions of Problem (P_C). Because of this property, the above game form guarantees to provide an optimal centralized allocation irrespective of which NE is achieved in the game induced by it.

The assertion of Theorem 1 that establishes the above two properties of the game form presented in Section III-A is based on the assumption that there exists a NE of the game induced by this game form and the users' utility functions (2). However, Theorem 1 does not say anything about the existence of NE. Theorem 2 asserts that NE exist in the above game, and provides conditions that characterize the set of all NE that result in optimal centralized allocations of the form $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}}) = (\hat{\boldsymbol{a}}^*_{\mathcal{N}}, (\sum_{j\in\mathcal{R}_i} l^*_{ij}\hat{a}^*_j)_{i\in\mathcal{N}})$, where $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ is any optimal centralized action profile. In addition to the above, Theorem 2 also establishes the following property of the game form. Since the optimal action profile $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ in the statement of Theorem 2 is arbitrary, the theorem implies that the game form of Section III-A can obtain each of the optimum action profiles of Problem (P_C) through at least one of the NE of the induced game. This establishes that the above game form is not biased towards any particular optimal centralized action profile.

We present the proofs of Theorems 1 and 2 in Appendices A and B. An example that illustrates how the properties established by Theorems 1 and 2 are achieved by the proposed game form can be found in [13].

IV. Future Directions

The problem formulation and the solution of the local public goods provisioning problem presented in this paper open up several new directions for future research. First, the development of efficient mechanisms that can compute NE is an important open problem. To address this problem there can be two different directions. (i) The development of algorithms that guarantee convergence to Nash equilibria of the games constructed in this paper. (ii) The development of alternative mechanisms/game forms that lead to games with dynamically stable NE. Second, the network model we studied in this paper assumed a given set of users and a given network topology. In many local public good networks such as social or research networks, the set of network users and the network topology must be determined as part of network objective maximization. These situations give rise to interesting admission control and network formation problems many of which are open research problems. Finally, in this paper we focused on static resource allocation problem where the characteristics of the local public good network do not change with time. The development of implementation mechanisms under dynamic situations, where the network characteristics change during the determination of resource allocation, are open research problems.

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In the appendices that follow, we present the proof of Theorems 1 and 2. We divide the proof into several claims to organize the presentation.

APPENDIX A
PROOF OF THEOREM 1

We prove Theorem 1 in four claims. In Claims 2 and 3 we show that all users weakly prefer a NE allocation (corresponding to the game form presented in Section III-A) to their initial allocations; these claims prove part (a) of Theorem 1. In Claim 1 we show that a NE allocation is a feasible solution of Problem (P_C). In Claim 4 we show that a NE action profile is an optimal action profile for Problem (P_C). Thus, Claim 1 and Claim 4 establish that a NE allocation is an optimal solution of Problem (P_C) and prove part (b) of Theorem 1.

Claim 1: If $\boldsymbol{m}^*_{\mathcal{N}}$ is a NE of the game induced by the game form presented in Section III-A and the users' utility functions (2), then the action and tax profile $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}}) := (\hat{\boldsymbol{a}}_{\mathcal{N}}(\boldsymbol{m}^*_{\mathcal{N}}), \hat{\boldsymbol{t}}_{\mathcal{N}}(\boldsymbol{m}^*_{\mathcal{N}}))$ is a feasible solution of Problem (P_C), i.e. $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}}) \in \mathcal{D}$.

Proof: We prove the feasibility of the NE action and tax profiles in two steps. First we prove the feasibility of the NE tax profile, then we prove the feasibility of the NE action profile.

To prove the feasibility of NE tax profile, we need to show that it satisfies (1). For this, we first add the second and third terms on the Right Hand Side (RHS) of (8) $\forall\, i \in \mathcal{N}$, i.e.

$$\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{R}_i}\left[{}^{i}\pi_j\left({}^{i}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j\right)^2 - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j \left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j\right)^2\right]. \tag{11}$$

From the construction of the graph matrix $\mathcal{G}$ and the sets $\mathcal{R}_i$ and $\mathcal{C}_j$, $i, j \in \mathcal{N}$, the sum $\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{R}_i}(\cdot)$ is equal to the sum $\sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{C}_j}(\cdot)$. Therefore, we can rewrite (11) as

$$\sum_{j\in\mathcal{N}}\left[\sum_{i\in\mathcal{C}_j}{}^{i}\pi_j\left({}^{i}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j\right)^2 - \sum_{i\in\mathcal{C}_j}{}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j \left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j\right)^2\right]. \tag{12}$$

Note that both the sums inside the square brackets in (12) are over all $i \in \mathcal{C}_j$. Because of the cyclic indexing of the users in each set $\mathcal{C}_j$, $j \in \mathcal{N}$, these two sums are equal. Therefore the overall sum in (12) evaluates to zero. Thus, the sum of taxes in (8) reduces to

$$\sum_{i\in\mathcal{N}}\hat{t}_i((\boldsymbol{m}_{\mathcal{C}_j})_{j\in\mathcal{R}_i}) = \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{R}_i} l_{ij}(\boldsymbol{m}_{\mathcal{C}_j})\,\hat{a}_j(\boldsymbol{m}_{\mathcal{C}_j}). \tag{13}$$

Combining (9) and (13) we obtain

$$\sum_{i\in\mathcal{N}}\hat{t}_i((\boldsymbol{m}_{\mathcal{C}_j})_{j\in\mathcal{R}_i}) = \sum_{j\in\mathcal{N}}\left[\sum_{i\in\mathcal{C}_j}{}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j - \sum_{i\in\mathcal{C}_j}{}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}\pi_j\right]\hat{a}_j(\boldsymbol{m}_{\mathcal{C}_j}) = 0. \tag{14}$$

The second equality in (14) follows because of the cyclic indexing of the users in each set $\mathcal{C}_j$, $j \in \mathcal{N}$, which makes the two sums inside the square brackets in (14) equal. Because (14) holds for any arbitrary message profile $\boldsymbol{m}_{\mathcal{N}}$, it follows that at NE $\boldsymbol{m}^*_{\mathcal{N}}$,

$$\sum_{i\in\mathcal{N}}\hat{t}_i((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}) = 0. \tag{15}$$

To complete the proof of Claim 1, we have to prove that for all $i \in \mathcal{N}$, $\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}) \in \mathcal{A}_i$. We prove this by contradiction. Suppose $\hat{a}^*_i \notin \mathcal{A}_i$ for some $i \in \mathcal{N}$. Then, from (2), $u^A_i(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i) = -\infty$. Consider $\widetilde{\boldsymbol{m}}_i = (({}^{i}\widetilde{a}_i, {}^{i}\boldsymbol{a}^*_{\mathcal{R}_i}/i), {}^{i}\boldsymbol{\pi}^*_{\mathcal{R}_i})$ where ${}^{i}a^*_k, k \in \mathcal{R}_i\backslash\{i\}$, and ${}^{i}\boldsymbol{\pi}^*_{\mathcal{R}_i}$ are respectively the NE action and price proposals of user i and ${}^{i}\widetilde{a}_i$ is such that

$$\hat{a}_i(\widetilde{\boldsymbol{m}}_i, \boldsymbol{m}^*_{\mathcal{C}_i}/i) = \frac{1}{|\mathcal{C}_i|}\Big({}^{i}\widetilde{\boldsymbol{a}}_i + \sum_{\substack{k\in\mathcal{C}_i\\k\neq i}}{}^{k}a^*_i\Big) \in \mathcal{A}_i. \tag{16}$$

Note that the flexibility of user i in choosing any message ${}^{i}\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}$ (see (5)) allows it to choose an appropriate ${}^{i}\widetilde{a}_i$ that satisfies the condition in (16). For the message $\widetilde{\boldsymbol{m}}_i$ constructed above,

$$\begin{aligned} &u^A_i\Big(\big(\hat{a}_k(\widetilde{\boldsymbol{m}}_i, \boldsymbol{m}^*_{\mathcal{C}_k}/i)\big)_{k\in\mathcal{R}_i},\;\; \hat{t}_i\big((\widetilde{\boldsymbol{m}}_i,\, \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big)\Big) = \\ &-\hat{t}_i\big((\widetilde{\boldsymbol{m}}_i,\, \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big) + u_i\Big(\big(\hat{a}_k(\widetilde{\boldsymbol{m}}_i, \boldsymbol{m}^*_{\mathcal{C}_k}/i)\big)_{k\in\mathcal{R}_i}\Big) \\ &> -\infty = u^A_i(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i). \end{aligned} \tag{17}$$

Thus if $\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}) \notin \mathcal{A}_i$ user i finds it profitable to deviate to $\widetilde{\boldsymbol{m}}_i$. Inequality (17) implies that $\boldsymbol{m}^*_{\mathcal{N}}$ cannot be a NE, which is a contradiction. Therefore, at any NE $\boldsymbol{m}^*_{\mathcal{N}}$, we must have $\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}) \in \mathcal{A}_i\ \forall\, i \in \mathcal{N}$. This along with (15) implies that, $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}}) \in \mathcal{D}$. □

Claim 2: If $\boldsymbol{m}^*_{\mathcal{N}}$ is a NE of the game induced by the game form presented in Section III-A and the users' utility functions (2), then, the tax $\hat{t}_i((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}) =: \hat{t}^*_i$ paid by user $i, i \in \mathcal{N}$, at the NE $\boldsymbol{m}^*_{\mathcal{N}}$ is of the form $\hat{t}^*_i = \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j$, where $l^*_{ij} = l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j})$ and $\hat{a}^*_j = \hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})$.

Proof: Let $\boldsymbol{m}^*_{\mathcal{N}}$ be the NE specified in the statement of Claim 2. Then, for each $i \in \mathcal{N}$,

$$\begin{aligned} &u^A_i\Big(\big(\hat{a}_k(\boldsymbol{m}_i, \boldsymbol{m}^*_{\mathcal{C}_k}/i)\big)_{k\in\mathcal{R}_i},\, \hat{t}_i\big((\boldsymbol{m}_i,\, \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big)\Big) \\ &\leq u^A_i\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i\Big),\ \forall\, \boldsymbol{m}_i \in \mathcal{M}_i. \end{aligned} \tag{18}$$

Substituting $\boldsymbol{m}_i = ({}^i\boldsymbol{a}^*_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i})$, ${}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}$, in (18) and using (7) implies that

$$u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i},\ \hat{t}_i\big((({}^i\boldsymbol{a}^*_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}), \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big)\Big) \leq\ u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}_i^*\Big), \quad \forall\, {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}. \tag{19}$$

Since u_i^A decreases in t_i (see (2)), (19) implies that

$$\hat{t}_i\Big((({}^i\boldsymbol{a}^*_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}), \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\Big) \geq \hat{t}_i^*,\ \ \forall\, {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}. \tag{20}$$

Substituting (8) in (20) results in

$$\begin{aligned}&\sum_{j\in\mathcal{R}_i}\Bigg[l^*_{ij}\hat{a}^*_j + {}^i\pi_j\left({}^ia^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a^*_j\right)^2 - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi^*_j \left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a^*_j\right)^2\Bigg] \geq \\ &\sum_{j\in\mathcal{R}_i}\Bigg[l^*_{ij}\hat{a}^*_j + {}^i\pi^*_j\left({}^ia^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a^*_j\right)^2 - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi^*_j \left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a^*_j\right)^2\Bigg],\ \forall\, {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}.\end{aligned} \tag{21}$$

Canceling the common terms in (21) gives

$$\sum_{j\in\mathcal{R}_i}({}^i\pi_j - {}^i\pi^*_j)\left({}^ia^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a^*_j\right)^2 \geq 0,\ \forall\, {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}. \tag{22}$$

Since (22) must hold for all ${}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}$, we must have that

$$\text{for each } j \in \mathcal{R}_i,\ \text{either } {}^i\pi^*_j = 0 \text{ or } {}^ia^*_j = {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a^*_j. \tag{23}$$

From (23) it follows that at any NE $\boldsymbol{m}^*_{\mathcal{N}}$,

$${}^i\pi^*_j\left({}^ia^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a^*_j\right)^2 = 0, \quad \forall\, j \in \mathcal{R}_i, \quad \forall\, i \in \mathcal{N}. \tag{24}$$

Note that (24) also implies that $\forall\, i \in \mathcal{N}$ and $\forall\, j \in \mathcal{R}_i$,

$${}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi^*_j\left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a^*_j\right)^2 = 0. \tag{25}$$

(25) follows from (24) because for each $i \in \mathcal{N}$, $j \in \mathcal{R}_i$ also implies that $j \in \mathcal{R}_{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}$. Using (24) and (25) in (8) we obtain that any NE tax profile must be of the form

$$\hat{t}^*_i = \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j, \quad \forall\, i \in \mathcal{N}. \tag{26}$$

□

Claim 3: The game form given in Section III-A is individually rational, i.e. at every NE $\boldsymbol{m}^*_{\mathcal{N}}$ of the game induced by this game form and the users' utilities in (2), each user $i \in \mathcal{N}$ weakly prefers the allocation $(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i)$ to the initial allocation $(\boldsymbol{0}, 0)$. Mathematically,

$$u_i^A\Big(\boldsymbol{0},\ 0\Big) \leq\ u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i\Big), \qquad \forall\, i \in \mathcal{N}. \tag{27}$$

Proof: Suppose $\boldsymbol{m}^*_{\mathcal{N}}$ is a NE of the game induced by the game form of Section III-A and the users' utility functions (2). From Claim 2 we know the form of users' tax at $\boldsymbol{m}^*_{\mathcal{N}}$. Substituting that from (26) into (18) we obtain that for each $i \in \mathcal{N}$,

$$u_i^A\Big(\big(\hat{a}_k(\boldsymbol{m}_i, \boldsymbol{m}^*_{\mathcal{C}_k}/i)\big)_{k\in\mathcal{R}_i},\ \ \hat{t}_i\big((\boldsymbol{m}_i, \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big)\Big) \leq u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j\Big),\ \forall\, \boldsymbol{m}_i = ({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}) \in \mathcal{M}_i. \tag{28}$$

Substituting for $\hat{t}_i$ in (28) from (8) and using (25) we obtain,

$$\begin{aligned}&u_i^A\Bigg(\Big(\hat{a}_k\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}), \boldsymbol{m}^*_{\mathcal{C}_k}/i\big)\Big)_{k\in\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i}\Big(l^*_{ij}\ \hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}), \boldsymbol{m}^*_{\mathcal{C}_j}/i\big) + {}^i\pi_j\big({}^ia_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j\big)^2\Big)\Bigg) \\ &\leq u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j\Big),\ \forall\, {}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|},\ \forall\, {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}.\end{aligned} \tag{29}$$

In particular, ${}^i\boldsymbol{\pi}_{\mathcal{R}_i} = \boldsymbol{0}$ in (29) implies that

$$u_i^A\Bigg(\Big(\hat{a}_k\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, \boldsymbol{0}), \boldsymbol{m}^*_{\mathcal{C}_k}/i\big)\Big)_{k\in\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i}\Big(l^*_{ij}\ \hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, \boldsymbol{0}), \boldsymbol{m}^*_{\mathcal{C}_j}/i\big)\Big)\Bigg) \leq\ u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j\Big),\ \forall\, {}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}. \tag{30}$$

Since (30) holds for all ${}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}$, substituting in it $\frac{1}{|\mathcal{C}_j|}({}^ia_j + \sum_{k\in\mathcal{C}_j\setminus\{i\}} {}^ka^*_j) = \overline{a}_j\ \forall\, j \in \mathcal{R}_i$ implies,

$$u_i^A\Big(\big(\overline{a}_j\big)_{j\in\mathcal{R}_i},\ \sum_{j\in\mathcal{R}_i}\big(l^*_{ij}\ \overline{a}_j\big)\Big) \leq\ u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j\Big),\quad \forall\, \overline{a}_{\mathcal{R}_i} := (\overline{a}_j)_{j\in\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}. \tag{31}$$

For $\overline{a}_{\mathcal{R}_i} = \boldsymbol{0}$, (31) implies further that

$$u_i^A\Big(\boldsymbol{0},\ 0\Big) \leq\ u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j\Big), \quad \forall\, i \in \mathcal{N}.$$

□

Claim 4: A NE allocation $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$ is an optimal solution of the centralized problem (P_C).

Proof: For each $i \in \mathcal{N}$, (31) can be equivalently written as

$$\begin{aligned}\hat{\boldsymbol{a}}^*_{\mathcal{R}_i} &\in \underset{\overline{\boldsymbol{a}}_{\mathcal{R}_i}\in\mathbb{R}^{|\mathcal{R}_i|}}{\arg\max}\ u_i^A\Big(\overline{\boldsymbol{a}}_{\mathcal{R}_i},\ \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \overline{a}_j\Big) \\ &= \underset{\substack{\overline{a}_i\in\mathcal{A}_i \\ \overline{a}_j\in\mathbb{R},\, j\in\mathcal{R}_i\setminus\{i\}}}{\arg\max}\ \left\{-\sum_{j\in\mathcal{R}_i} l^*_{ij}\ \overline{a}_j + u_i(\overline{\boldsymbol{a}}_{\mathcal{R}_i})\right\}\end{aligned} \tag{32}$$

Let for each $i \in \mathcal{N}$, $f_{\mathcal{A}_i}(a_i)$ be a convex function that characterizes the set $\mathcal{A}_i$ as, $a_i \in \mathcal{A}_i \Leftrightarrow f_{\mathcal{A}_i}(a_i) \leq 0$. [11]

Since for each $i \in \mathcal{N}$, $u_i(\overline{\boldsymbol{a}}_{\mathcal{R}_i})$ is assumed to be concave in $\overline{\boldsymbol{a}}_{\mathcal{R}_i}$ and $\mathcal{A}_i$ is convex, the Karush Kuhn Tucker (KKT) conditions [20, Chapter 11] are necessary and sufficient for $\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}$ to be a maximizer in (32). Thus, for each $i \in \mathcal{N}\ \exists\ \lambda_i \in \mathbb{R}_+$ such that, $\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}$ and λ_i satisfy the following KKT conditions:

$$\begin{aligned}&\forall\, j \in \mathcal{R}_i\setminus\{i\}, \quad l^*_{ij} - \nabla_{\overline{a}_j} u_i(\overline{\boldsymbol{a}}_{\mathcal{R}_i})\ |_{\overline{\boldsymbol{a}}_{\mathcal{R}_i}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}} = 0, \\ &l^*_{ii} - \nabla_{\overline{a}_i} u_i(\overline{\boldsymbol{a}}_{\mathcal{R}_i})\ |_{\overline{\boldsymbol{a}}_{\mathcal{R}_i}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}} + \lambda_i \nabla_{\overline{a}_i} f_{\mathcal{A}_i}(\overline{a}_i)\ |_{\overline{a}_i=\hat{a}^*_i} = 0, \\ &\lambda_i f_{\mathcal{A}_i}(\hat{a}^*_i) = 0.\end{aligned} \tag{33}$$

For each $i \in \mathcal{N}$, adding the KKT condition equations in (33) over $k \in \mathcal{C}_i$ results in

$$\sum_{k\in\mathcal{C}_i} l^*_{ki} - \nabla_{\overline{a}_i}\sum_{k\in\mathcal{C}_i} u_k(\overline{\boldsymbol{a}}_{\mathcal{R}_k})\ |_{\overline{\boldsymbol{a}}_{\mathcal{R}_k}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_k}} + \lambda_i \nabla_{\overline{a}_i} f_{\mathcal{A}_i}(\overline{a}_i)\ |_{\overline{a}_i=\hat{a}^*_i}\ =\ 0. \tag{34}$$

[11] By [20] we can find a convex function that characterizes a convex set.

From (9) we have,

$$\sum_{k\in\mathcal{C}_i} l^*_{ki} = \sum_{k\in\mathcal{C}_i} \left({}^{\mathcal{C}_{i(\mathcal{I}_{ki}+1)}}\pi^*_i - {}^{\mathcal{C}_{i(\mathcal{I}_{ki}+2)}}\pi^*_i \right) = 0. \quad (35)$$

Substituting (35) in (34) we obtain [12] $\forall\, i \in \mathcal{N}$,

$$\begin{aligned} -\nabla_{\overline{a}_i} \sum_{k\in\mathcal{C}_i} u_k(\overline{\boldsymbol{a}}_{\mathcal{R}_k})\,|_{\overline{\boldsymbol{a}}_{\mathcal{R}_k}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_k}} + \lambda_i \nabla_{\overline{a}_i} f_{\mathcal{A}_i}(\overline{a}_i)\,|_{\overline{a}_i=\hat{a}^*_i} &= 0, \\ \lambda_i f_{\mathcal{A}_i}(\hat{a}^*_i) &= 0. \end{aligned} \quad (36)$$

The conditions in (36) along with the non-negativity of $\lambda_i, i \in \mathcal{N}$, specify the KKT conditions (for variable $\hat{\boldsymbol{a}}_{\mathcal{N}}$) for Problem (P_C). Since (P_C) is a concave optimization problem, KKT conditions are necessary and sufficient for optimality. As shown in (36), the action profile $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ satisfies these optimality conditions. Furthermore, the tax profile $\hat{\boldsymbol{t}}^*_{\mathcal{N}}$ satisfies, by its definition, $\sum_{i\in\mathcal{N}} \hat{t}^*_i = 0$. Therefore, the NE allocation $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$ is an optimal solution of Problem (P_C). This completes the proof of Claim 4 and hence, the proof of Theorem 1. □

Claims 1–4 (Theorem 1) establish the properties of NE allocations based on the assumption that there exists a NE of the game induced by the game form of Section III-A and users' utility functions (2). However, these claims do not guarantee the existence of a NE. This is guaranteed by Theorem 2 which is proved next in Claims 5 and 6.

Appendix B
Proof of Theorem 2

We prove Theorem 2 in two steps. In the first step we show that if the centralized problem (P_C) has an optimal action profile $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$, there exist a set of personalized prices, one for each user $i \in \mathcal{N}$, such that when each $i \in \mathcal{N}$ individually maximizes its own utility taking these prices as given, it obtains $\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}$ as an optimal action profile. In the second step we show that the optimal action profile $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ and the corresponding personalized prices can be used to construct message profiles that are NE of the game induced by the game form of Section III-A and users' utility functions in (2).

Claim 5: If Problem (P_C) has an optimal action profile $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$, there exist a set of personalized prices $l^*_{ij}, j \in \mathcal{R}_i, i \in \mathcal{N}$, s.t.

$$\hat{\boldsymbol{a}}^*_{\mathcal{R}_i} \in \underset{\substack{\hat{a}_i\in\mathcal{A}_i \\ \hat{a}_j\in\mathbb{R},\, j\in\mathcal{R}_i\setminus\{i\}}}{\arg\max} \; -\sum_{j\in\mathcal{R}_i} l^*_{ij}\,\hat{a}_j + u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}), \;\; \forall\, i \in \mathcal{N}. \quad (37)$$

Proof: Suppose $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ is an optimal action profile corresponding to Problem (P_C). Writing the optimization problem (P_C) only in terms of variable $\hat{\boldsymbol{a}}_{\mathcal{N}}$ gives

$$\begin{aligned} \hat{\boldsymbol{a}}^*_{\mathcal{N}} \in \;& \underset{\hat{\boldsymbol{a}}_{\mathcal{N}}}{\arg\max} \sum_{i\in\mathcal{N}} u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}) \\ \text{s.t.}\;\; & \hat{a}_i \in \mathcal{A}_i,\; \forall\, i \in \mathcal{N}. \end{aligned} \quad (38)$$

As stated earlier, an optimal solution of Problem (P_C) is of the form $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}_{\mathcal{N}})$, where $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ is a solution of (38) and $\hat{\boldsymbol{t}}_{\mathcal{N}} \in \mathbb{R}^N$ is any tax profile that satisfies (1). Because KKT conditions are necessary for optimality, the optimal solution in (38) must satisfy the KKT conditions. This implies that there exist $\lambda_i \in \mathbb{R}_+, i \in \mathcal{N}$, such that for each $i \in \mathcal{N}$, λ_i and $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ satisfy

$$\begin{aligned} -\nabla_{\hat{a}_i} \sum_{k\in\mathcal{C}_i} u_k(\hat{\boldsymbol{a}}_{\mathcal{R}_k})\,|_{\hat{\boldsymbol{a}}_{\mathcal{R}_k}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_k}} + \lambda_i \nabla_{\hat{a}_i} f_{\mathcal{A}_i}(\hat{a}_i)\,|_{\hat{a}_i=\hat{a}^*_i} &= 0, \\ \lambda_i f_{\mathcal{A}_i}(\hat{a}^*_i) &= 0, \end{aligned} \quad (39)$$

where $f_{\mathcal{A}_i}(\cdot)$ is the convex function defined in Claim 4. Defining for each $i \in \mathcal{N}$,

$$\begin{aligned} l^*_{ij} &:= \nabla_{\hat{a}_j} u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i})\,|_{\hat{\boldsymbol{a}}_{\mathcal{R}_i}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}}, \qquad j \in \mathcal{R}_i\setminus\{i\}, \\ l^*_{ii} &:= \nabla_{\hat{a}_i} u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i})\,|_{\hat{\boldsymbol{a}}_{\mathcal{R}_i}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}} - \lambda_i \nabla_{\hat{a}_i} f_{\mathcal{A}_i}(\hat{a}_i)\,|_{\hat{a}_i=\hat{a}^*_i}, \end{aligned} \quad (40)$$

we get $\forall\, i \in \mathcal{N}$,

$$\begin{aligned} &\sum_{k\in\mathcal{C}_i} l^*_{ki} = \\ &\nabla_{\hat{a}_i} \sum_{k\in\mathcal{C}_i} u_k(\hat{\boldsymbol{a}}_{\mathcal{R}_k})\,|_{\hat{\boldsymbol{a}}_{\mathcal{R}_k}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_k}} - \lambda_i \nabla_{\hat{a}_i} f_{\mathcal{A}_i}(\hat{a}_i)\,|_{\hat{a}_i=\hat{a}^*_i} = 0. \end{aligned} \quad (41)$$

The second equality in (41) follows from (39). Furthermore, (40) implies that $\forall\, i \in \mathcal{N}$,

$$\begin{aligned} \forall\, j \in \mathcal{R}_i\setminus\{i\}, \quad l^*_{ij} - \nabla_{\hat{a}_j} u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i})\,|_{\hat{\boldsymbol{a}}_{\mathcal{R}_i}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}} &= 0, \\ l^*_{ii} - \nabla_{\hat{a}_i} u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i})\,|_{\hat{\boldsymbol{a}}_{\mathcal{R}_i}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}} + \lambda_i \nabla_{\hat{a}_i} f_{\mathcal{A}_i}(\hat{a}_i)\,|_{\hat{a}_i=\hat{a}^*_i} &= 0. \end{aligned} \quad (42)$$

The equations in (42) along with the second equality in (39) imply that for each $i \in \mathcal{N}$, $\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}$ and λ_i satisfy the KKT conditions for the following maximization problem:

$$\max_{\substack{\hat{a}_i\in\mathcal{A}_i \\ \hat{a}_j\in\mathbb{R},\, j\in\mathcal{R}_i\setminus\{i\}}} \; -\sum_{j\in\mathcal{R}_i} l^*_{ij}\,\hat{a}_j + u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}) \quad (43)$$

Because the objective function in (43) is concave (Assumption 4), KKT conditions are necessary and sufficient for optimality. Therefore, we conclude from (42) and (39) that,

$$\hat{\boldsymbol{a}}^*_{\mathcal{R}_i} \in \underset{\substack{\hat{a}_i\in\mathcal{A}_i \\ \hat{a}_j\in\mathbb{R},\, j\in\mathcal{R}_i\setminus\{i\}}}{\arg\max} \; -\sum_{j\in\mathcal{R}_i} l^*_{ij}\,\hat{a}_j + u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}), \quad \forall\, i \in \mathcal{N}. \;\square$$

Claim 6: Let $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ be an optimal action profile for Problem (P_C), let $l^*_{ij}, j \in \mathcal{R}_i, i \in \mathcal{N}$, be the personalized prices corresponding to $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ as defined in Claim 5, and let $\hat{t}^*_i := \sum_{j\in\mathcal{R}_i} l^*_{ij}\hat{a}^*_j, i \in \mathcal{N}$. Let $\boldsymbol{m}^*_i := ({}^i\boldsymbol{a}^*_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}^*_{\mathcal{R}_i}), i \in \mathcal{N}$, be a solution to the following set of relations:

$$\begin{aligned} \frac{1}{|\mathcal{C}_i|} \sum_{k\in\mathcal{C}_i} {}^k a^*_i \;&=\; \hat{a}^*_i, \quad i \in \mathcal{N}, && (44) \\ {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}\pi^*_j \;&=\; l^*_{ij}, \quad j \in \mathcal{R}_i,\; i \in \mathcal{N}, && (45) \\ {}^i\pi^*_j \left({}^i a^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}} a^*_j \right)^2 \;&=\; 0, \quad j \in \mathcal{R}_i,\; i \in \mathcal{N}, && (46) \\ {}^i\pi^*_j \;&\geq\; 0, \quad j \in \mathcal{R}_i,\; i \in \mathcal{N}. && (47) \end{aligned}$$

Then, $\boldsymbol{m}^*_{\mathcal{N}} := (\boldsymbol{m}^*_1, \boldsymbol{m}^*_2, \ldots, \boldsymbol{m}^*_N)$ is a NE of the game induced by the game form of Section III-A and the users' utility functions (2). Furthermore, for each $i \in \mathcal{N}$, $\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}) = \hat{a}^*_i$, $l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j}) = l^*_{ij}, j \in \mathcal{R}_i$, and $\hat{t}_i((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}) = \hat{t}^*_i$.

Proof: Note that, the conditions in (44)–(47) are necessary for any NE $\boldsymbol{m}^*_{\mathcal{N}}$ of the game induced by the game form of Section III-A and users' utilities (2), to result in the allocation $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$ (see (7), (9) and (24)). Therefore, the set of solutions of (44)–(47), if such a set exists, is a superset of the set of all NE corresponding to the above game that result in $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$.

[12] The second equality in (36) is one of the KKT conditions from (33).

Below we show that the solution set of (44)–(47) is in fact exactly the set of all NE that result in $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$.

To prove this, we first show that the set of relations in (44)–(47) do have a solution. Notice that (44) and (46) are satisfied by setting for each $i \in \mathcal{N}$, ${}^k a_i^* = \hat{a}_i^* \ \forall\ k \in \mathcal{C}_i$. Notice also that for each $j \in \mathcal{N}$, the sum over $i \in \mathcal{C}_j$ of the right hand side of (45) is 0. Therefore, for each $j \in \mathcal{N}$, (45) has a solution in ${}^i\pi_j^*, i \in \mathcal{C}_j$. Furthermore, for any solution ${}^i\pi_j^*, i \in \mathcal{C}_j, j \in \mathcal{N}$, of (45), ${}^i\pi_j^* + c, i \in \mathcal{C}_j, j \in \mathcal{N}$, where c is some constant, is also a solution of (45). Consequently, by appropriately choosing c, we can select a solution of (45) such that (47) is satisfied.

It is clear from the above discussion that (44)–(47) have multiple solutions. We now show that the set of solutions $\boldsymbol{m}^*_{\mathcal{N}}$ of (44)–(47) is the set of NE that result in $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$. From Claim 5, (37) can be equivalently written as

$$\hat{\boldsymbol{a}}^*_{\mathcal{R}_i} \in \underset{\hat{\boldsymbol{a}}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}}{\arg\max}\ u_i^A\Big(\hat{\boldsymbol{a}}_{\mathcal{R}_i}, \sum_{j \in \mathcal{R}_i} l_{ij}^*\ \hat{a}_j\Big), \qquad i \in \mathcal{N}. \tag{48}$$

Substituting $\hat{a}_j|\mathcal{C}_j| - \sum_{k \in \mathcal{C}_j \backslash \{i\}} {}^k a_j^* = {}^i a_j$ for each $j \in \mathcal{R}_i$, $i \in \mathcal{N}$, in (48) we obtain

$$\begin{aligned} {}^i\boldsymbol{a}^*_{\mathcal{R}_i} \in \underset{{}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}}{\arg\max}\ u_i^A \bigg(&\Big(\frac{1}{|\mathcal{C}_j|}\big({}^i a_j + \sum_{k \in \mathcal{C}_j \backslash \{i\}} {}^k a_j^*\big)\Big)_{j \in \mathcal{R}_i}, \\ &\sum_{j \in \mathcal{R}_i} l_{ij}^*\ \frac{1}{|\mathcal{C}_j|}\big({}^i a_j + \sum_{k \in \mathcal{C}_j \backslash \{i\}} {}^k a_j^*\big)\bigg),\ i \in \mathcal{N}. \end{aligned} \tag{49}$$

Because of (46), (49) also implies that

$$\begin{aligned} &({}^i\boldsymbol{a}^*_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}^*_{\mathcal{R}_i}) \in \\ &\underset{({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}) \in \mathbb{R}^{|\mathcal{R}_i|} \times \mathbb{R}_+^{|\mathcal{R}_i|}}{\arg\max}\ u_i^A \bigg(\Big(\hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}),\ \boldsymbol{m}^*_{\mathcal{C}_j}/i\big)\Big)_{j \in \mathcal{R}_i}, \\ &\sum_{j \in \mathcal{R}_i} l_{ij}^*\ \hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}),\ \boldsymbol{m}^*_{\mathcal{C}_j}/i\big) - \sum_{j \in \mathcal{R}_i} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j^* \\ &\Big({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}} a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}} a_j^*\Big)^2 \bigg), \qquad i \in \mathcal{N}. \end{aligned} \tag{50}$$

Furthermore, since u_i^A is strictly decreasing in the tax (see (2)), (50) also implies the following:

$$\begin{aligned} &({}^i\boldsymbol{a}^*_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}^*_{\mathcal{R}_i}) \in \\ &\underset{({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}) \in \mathbb{R}^{|\mathcal{R}_i|} \times \mathbb{R}_+^{|\mathcal{R}_i|}}{\arg\max}\ u_i^A \bigg(\Big(\hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}), \boldsymbol{m}^*_{\mathcal{C}_j}/i\big)\Big)_{j \in \mathcal{R}_i}, \\ &\sum_{j \in \mathcal{R}_i} l_{ij}^* \hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}), \boldsymbol{m}^*_{\mathcal{C}_j}/i\big) + \sum_{j \in \mathcal{R}_i} {}^i\pi_j \Big({}^i a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}} a_j^*\Big)^2 \\ &- \sum_{j \in \mathcal{R}_i} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j^* \Big({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}} a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}} a_j^*\Big)^2 \bigg), \qquad i \in \mathcal{N}. \end{aligned} \tag{51}$$

Eq. (51) implies that, if the message exchange and allocation is done according to the game form of Section III-A, then user $i, i \in \mathcal{N}$, maximizes its utility at $\boldsymbol{m}_i^*$ when all other users $j \in \mathcal{N} \backslash \{i\}$ choose their respective messages $\boldsymbol{m}_j^*, j \in \mathcal{N} \backslash \{i\}$. This, in turn, implies that a message profile $\boldsymbol{m}^*_{\mathcal{N}}$ that is a solution to (44)–(47) is a NE of the game induced by the above game form and the users' utilities (2). Furthermore, it follows from (44)–(47) that the allocation at $\boldsymbol{m}^*_{\mathcal{N}}$ is

$$\begin{aligned} \hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}) &= \frac{1}{|\mathcal{C}_i|} \sum_{k \in \mathcal{C}_i} {}^k a_i^* \ = \ \hat{a}_i^*, \quad i \in \mathcal{N}, \\ l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j}) &= {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}\pi_j^* = l_{ij}^*,\ j \in \mathcal{R}_i,\ i \in \mathcal{N}, \\ \hat{t}_i\big((\boldsymbol{m}^*_{\mathcal{C}_j})_{j \in \mathcal{R}_i}\big) &= \sum_{j \in \mathcal{R}_i} l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j}) \hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j}) \\ &\quad + {}^i\pi_j^* \Big({}^i a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}} a_j^*\Big)^2 \\ &\quad - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j^* \Big({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}} a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}} a_j^*\Big)^2 \\ &= \sum_{j \in \mathcal{R}_i} l_{ij}^* \hat{a}_i^* \ = \ \hat{t}_i^*, \quad i \in \mathcal{N}. \end{aligned} \tag{52}$$

From (52) it follows that the set of solutions $\boldsymbol{m}^*_{\mathcal{N}}$ of (44)–(47) is exactly the set of NE that result in $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$. This completes the proof of Claim 6 and hence the proof of Theorem 2. □

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